MAT 530: Topology&Geometry, I
Fall 2005

Notes on Van Kampen's Theorem

1 Summary

Van Kampen's Theorem is in a sense a pasting lemma for the fundamental group. It reconstructs the fundamental group of a topological space X from the fundamental groups of two open subsets, $\mathcal{U}_1$ and $\mathcal{U}_2$, that together make up X and of their intersection.

Van Kampen's Theorem can be stated in two ways. The first version simply states that the commutative diagram in Figure 1 is an amalgated product; see Definition 3. By Proposition 4, this implies that $\pi_1(X,p)$ and the homomorphisms j_1 and j_2 are uniquely determined by the homomorphisms i_1 and i_2 (up to isomorphism commuting with i_1 and i_2, as in Figure 4). The second version describes $\pi_1(X,p)$ in terms of the homomorphisms i_1 and i_2 directly. It is more suitable for applications, but the first version is actually easier to prove. Proposition 4 implies that the two versions are equivalent.

Theorem 1 (Van Kampen's Theorem) *Suppose X is a topological space, $\mathcal{U}_1,\mathcal{U}_2\subset X$ are path-connected open subsets such that $\mathcal{U}_1\cap\mathcal{U}_2$ is path-connected, and $p\in\mathcal{U}_1\cap\mathcal{U}_2$. If*

$$\begin{array}{ccccc} & & \pi_1(\mathcal{U}_1,p) & & \\ & \overset{i_1}{\nearrow} & & \overset{j_1}{\searrow} & \\ \pi_1(\mathcal{U}_1\cap\mathcal{U}_2,p) & & & & \pi_1(X,p) \\ & \underset{i_2}{\searrow} & & \underset{j_2}{\nearrow} & \\ & & \pi_1(\mathcal{U}_2,p) & & \end{array}$$

Figure 1: Van Kampen's Theorem Setting

are the group homomorphisms induced by the inclusions, then Figure 1 is an amalgated product. Thus, the homomorphism

$$j_1*j_2\colon \pi_1(\mathcal{U}_1,p)*\pi_1(\mathcal{U}_2,p) \longrightarrow \pi_1(X,p)$$

induced by j_1 and j_2 is surjective and $\ker j_1*j_2$ *is the least normal subgroup N of $\pi_1(\mathcal{U}_1,p)*\pi_1(\mathcal{U}_2,p)$ containing the set*

$$\big\{i_1(\alpha^{-1})i_2(\alpha)\colon \alpha\in\pi_1(\mathcal{U}_1\cap\mathcal{U}_2,p)\big\} \subset \pi_1(\mathcal{U}_1,p)*\pi_1(\mathcal{U}_2,p).$$

Thus,

$$\pi_1(X,p) \approx \pi_1(\mathcal{U}_1,p)*\pi_1(\mathcal{U}_2,p)/N.$$

Corollary 2 *Suppose X, $\mathcal{U}_1$, $\mathcal{U}_2$, p, i_1, i_2, j_1, and j_2 as are in Theorem 1.*
(1) If the homomorphisms i_1 and i_2 are trivial, then the homomorphism

$$j_1*j_2\colon \pi_1(\mathcal{U}_1,p)*\pi_1(\mathcal{U}_2,p) \longrightarrow \pi_1(X,p)$$

induced by j_1 and j_2 is an isomorphism.
(2) If the group $\pi_1(\mathcal{U}_2, p)$ is trivial, the homomorphism

$$j_1 \colon \pi_1(\mathcal{U}_1, p) \longrightarrow \pi_1(X, p)$$

is surjective and $\ker j_1$ *is the least normal subgroup of* $\pi_1(\mathcal{U}_1, p)$ *containing* $\mathrm{Im}\, i_1$.

2 Amalgated Product

Definition 3 specifies an object, $(\tilde{G}, j_1, j_2)$, by its properties instead of describing it directly. There are many such definitions in algebra (direct sum, free product), topology (subspace, product, and quotient topologies, covering map corresponding to a fixed conjugacy class, classifying spaces for groups), and geometry (moduli spaces of geometric objects, such as curves, surfaces, maps). Whenever such a definition is given one needs to show that an object with the desired properties exists and is unique. The latter is usually the easy part; in many cases (direct sum, free product, covering map) the proof is analogous to that in Proposition 4. On the other hand, showing that the object exists at all can be quite tricky and the argument is usually of a rather specific nature.

Definition 3 *A commutative diagram of group homomorphisms*

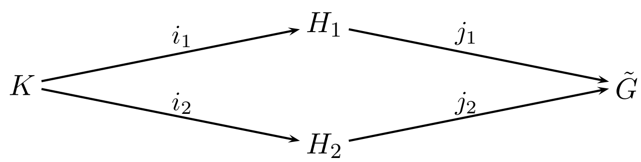


Figure 2: A Commutative Diagram

is an `amalgated product` *if for every group G and group homomorphisms $\phi_m : H_m \longrightarrow G$ for $m\!=\!1,2$ such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$, i.e. the solid arrows in the diagram*

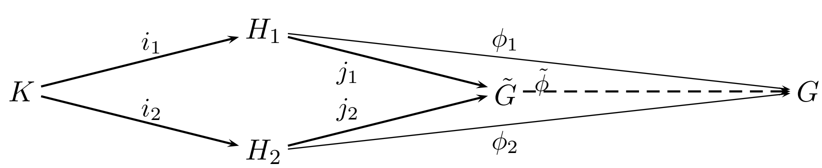


Figure 3: An Amalgated Product

commute, there exists a unique homomorphism $\tilde{\phi} : \tilde{G} \longrightarrow G$ such that $\phi_m = \phi \circ j_m$ for $m = 1, 2$, i.e. the entire diagram in Figure 3 commutes.

Proposition 4 *Suppose $i_1 \colon K \longrightarrow H_1$ and $i_2 \colon K \longrightarrow H_2$ are group homomorphisms. Let*

$$\tilde{G} = H_1 * H_2 / N,$$

where N is the smallest normal subgroup of $H_1 * H_2$ containing the set

$$\left\{ i_1(k^{-1}) i_2(k) \colon k \in K \right\} \subset H_1 * H_2.$$

If the homomorphisms $j_m\colon H_m \longrightarrow \tilde{G}$ *are defined by*

$$j_m(h) = hN \qquad \textit{for} \qquad m = 1, 2,$$

then the diagram in Figure 2 is an amalgated product. If Figure 2 with $(\tilde{G}, j_1, j_2)$ *replaced by another triple* $(\tilde{G}', j_1', j_2')$ *is an amalgated product, then there exists an isomorphism* $\tilde{G} \longrightarrow \tilde{G}'$ *(in fact, a unique one) such that the entire diagram*

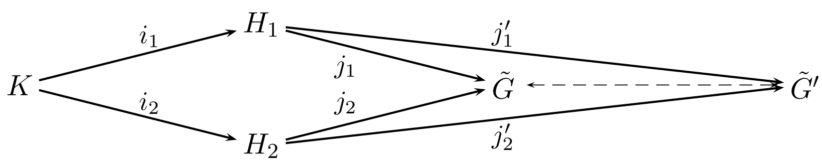


Figure 4: An Amalgated Product

commutes.

We first note that if Figure 2 is an amalgated product and $h\colon \tilde{G} \longrightarrow \tilde{G}$ is a homomorphism such that $h \circ j_m = j_m$ for $m = 1, 2$, i.e. the diagram

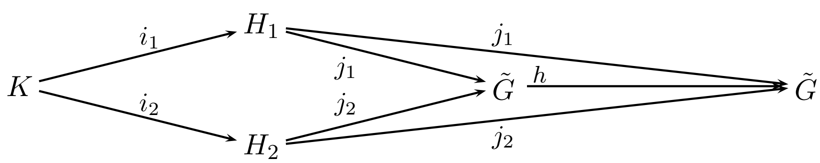


Figure 5: An Endomorphism of an Amalgated Product

commutes, then $h = \mathrm{id}_{\tilde{G}}$. This is immediate from the uniqueness of $\tilde{\phi}$ in Figure 3 in the case $(G, \phi_1, \phi_2) = (\tilde{G}, j_1, j_2)$. In other words, since $h = \mathrm{id}_{\tilde{G}}$ makes the diagram in Figure 5 commute, $\mathrm{id}_{\tilde{G}}$ is the only homomorphism with this property.

Suppose Figure 2 with $(\tilde{G}, j_1, j_2)$ replaced by another triple $(\tilde{G}', j_1', j_2')$ is an amalgated product. By Figure 3 with $(G, \phi_1, \phi_2) = (\tilde{G}', j_1', j_2')$, there exists a (unique) homomorphism

$$\tilde{\phi}\colon \tilde{G} \longrightarrow \tilde{G}' \qquad \text{s.t.} \qquad j_1' = \tilde{\phi} \circ j_1, \quad j_2' = \tilde{\phi} \circ j_2.$$

By Figure 3 with $(\tilde{G}, j_1, j_2)$ replaced by $(\tilde{G}', j_1', j_2')$ and $(G, \phi_1, \phi_2) = (\tilde{G}, j_1, j_2)$, there exists a (unique) homomorphism

$$\tilde{\phi}'\colon \tilde{G}' \longrightarrow \tilde{G} \qquad \text{s.t.} \qquad j_1 = \tilde{\phi}' \circ j_1', \quad j_2 = \tilde{\phi}' \circ j_2'.$$

Then,

$$\tilde{\phi}' \circ \tilde{\phi}\colon \tilde{G} \longrightarrow \tilde{G}' \qquad \text{and} \qquad \tilde{\phi} \circ \tilde{\phi}'\colon \tilde{G}' \longrightarrow \tilde{G}$$

are homomorphisms such that

$$\tilde{\phi}' \circ \tilde{\phi} \circ j_m = j_m \;\text{ for }\; m = 1, 2 \qquad \text{and} \qquad \tilde{\phi} \circ \tilde{\phi}' \circ j_m' = j_m' \;\text{ for }\; m = 1, 2.$$

By the previous paragraph, $\tilde{\phi}' \circ \tilde{\phi} = \mathrm{id}_{\tilde{G}}$ and $\tilde{\phi} \circ \tilde{\phi}' = \mathrm{id}_{\tilde{G}'}$. In other words, the homomorphism $\tilde{\phi}\colon \tilde{G} \longrightarrow \tilde{G}'$ is an isomorphism of amalgated products, i.e. the diagram

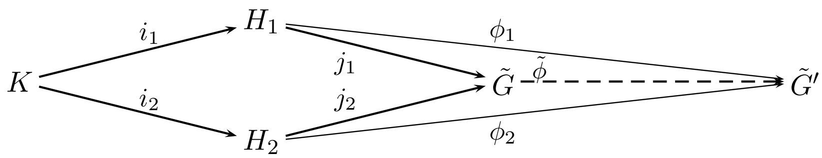


Figure 6: Isomorphism of Amalgated Products

commutes. This implies the second claim of Proposition 4.

Suppose next that $\tilde{G}=H_1*H_2/N$, j_1, and j_2 are as in the first part of Proposition 4. If $k\in K$, then

$$j_1\big(i_1(k)\big) = i_1(k)N = i_2(k)\big(i_1(k)^{-1}i_2(k)\big)^{-1}N = i_2(k)N = j_2\big(i_2(k)\big).$$

Thus, $j_1\circ i_1=j_2\circ i_2$, i.e. the diagram in Figure 2 is commutative as required.

Suppose in addition that G is a group and $\phi_m\colon H_m\longrightarrow G$ for $m=1,2$ is a group homomorphism such that $\phi_1\circ i_1=\phi_2\circ i_2$, i.e. the solid arrows in Figure 3 commute. We show that there exists a unique homomorphism $\tilde{\phi}\colon \tilde{G}\longrightarrow G$ such that $\phi_m=\tilde{\phi}\circ j_m$ for $m=1,2$, i.e. the entire diagram in Figure 3 is commutative. The conditions $\phi_m=\tilde{\phi}\circ j_m$ for $m=1,2$, determine $\tilde{\phi}$ on $\mathrm{Im}\, j_1$ and $\mathrm{Im}\, j_2$. Since $\mathrm{Im}\, j_1$ and $\mathrm{Im}\, j_2$ generate $\tilde{G}$, i.e. no proper subgroup of $\tilde{G}$ contains $\mathrm{Im}\, j_1$ and $\mathrm{Im}\, j_2$, it follows $\tilde{\phi}$ is unique, if it exists at all. Let

$$\phi_1*\phi_2\colon H_1*H_2\longrightarrow G$$

be the homomorphism induced by ϕ_1 and ϕ_2. Since $\phi_1\circ i_1=\phi_2\circ i_2$,

$$\begin{aligned}\phi_1*\phi_2\big(i_1(k^{-1})i_2(k)\big) = \phi_1\big(i_1(k^{-1})\big)\phi_2\big(i_2(k)\big) = \phi_1\big(i_1(k)\big)^{-1}\phi_2\big(i_2(k)\big) = 1\in G \quad \forall\, k\in K\\ \Longrightarrow \qquad i_1(k^{-1})i_2(k)\in \ker\phi_1*\phi_2 \quad \forall\, k\in K.\end{aligned}$$

Since $\ker\phi_1*\phi_2$ is a normal subgroup of H_1*H_2, $\ker\phi_1*\phi_2$ contains N. Thus, $\phi_1*\phi_2$ induces a homomorphism

$$\tilde{\phi}\colon \tilde{G}=H_1*H_2/N\longrightarrow G.$$

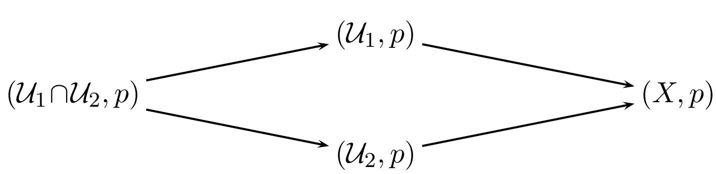
Furthermore,

$$\tilde{\phi}\big(j_m(h)\big) = \tilde{\phi}(hN) = \phi_1*\phi_2(h) = \phi_m(h) \quad \forall\, h\in H_m,\ m=1,2,$$

as needed.

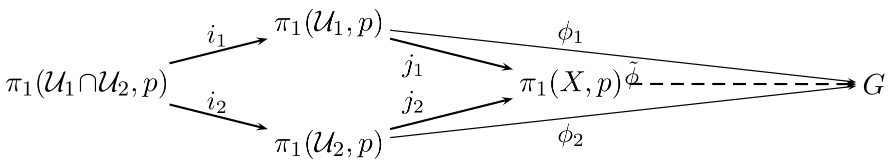
3 Proof of Van Kampen's Theorem

We show that the diagram in Figure 1 is an amalgated product. First, since the diagram



of inclusions is commutative, so is the diagram in Figure 1.

Suppose that G is a group and $\phi_m : \pi(\mathcal{U}_m, p) \longrightarrow G$ for $m=1,2$ is a group homomorphism such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$, i.e. the solid arrows in the diagram



commute. We show that there exists a unique homomorphism $\tilde{\phi} \colon \pi_1(X,p) \longrightarrow G$ such that $\phi_m = \tilde{\phi} \circ j_m$ for $m=1,2$, i.e. the entire diagram is commutative. The conditions $\phi_m = \tilde{\phi} \circ j_m$ for $m=1,2$ determine $\tilde{\phi}$ on $\operatorname{Im} j_1$ and $\operatorname{Im} j_2$. Since $\operatorname{Im} j_1$ and $\operatorname{Im} j_2$ generate $\pi_1(X,p)$, as will be re-proved below, it follows that $\tilde{\phi}$ is unique, if it exists. In the rest of this section we construct $\tilde{\phi}$.

Let $\mathcal{L}(\mathcal{U}_1 \cap \mathcal{U}_2, p)$, $\mathcal{L}(\mathcal{U}_1, p)$, $\mathcal{L}(\mathcal{U}_2, p)$, and $\mathcal{L}(X, p)$ denote the spaces of *loops* (not path-homotopy classes of loops) based at x_0 that are contained in $\mathcal{U}_1 \cap \mathcal{U}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, and X, respectively. If α is an element of $\mathcal{L}(\mathcal{U}_1 \cap \mathcal{U}_2, p)$, $\mathcal{L}(\mathcal{U}_1, p)$, $\mathcal{L}(\mathcal{U}_2, p)$, and $\mathcal{L}(X, p)$, its path-homotopy class in $\mathcal{U}_1 \cap \mathcal{U}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, and X, respectively, will be denoted by $[\alpha]_{\mathcal{U}_1 \cap \mathcal{U}_2}$, $[\alpha]_{\mathcal{U}_1}$, $[\alpha]_{\mathcal{U}_2}$, and $[\alpha]$, respectively. We will construct a map

$$\tilde{\Phi} \colon \mathcal{L}(X,p) \longrightarrow G$$

such that

($\tilde{\Phi}$1) $[\alpha]=[\beta] \Longrightarrow \tilde{\Phi}(\alpha)=\tilde{\Phi}(\beta)$ for all $\alpha, \beta \in \mathcal{L}(X,p)$;
($\tilde{\Phi}$2) $\tilde{\Phi}(\alpha * \beta)=\tilde{\Phi}(\alpha)\tilde{\Phi}(\beta)$ for all $\alpha, \beta \in \mathcal{L}(X,p)$;
($\tilde{\Phi}$3) $\tilde{\Phi}(\alpha)=\phi_m\big([\alpha]_{\mathcal{U}_m}\big)$ for all $\alpha \in \mathcal{L}(\mathcal{U}_m, p)$, $m=1,2$.

Once such a map is constructed, we can define

$$\tilde{\phi} \colon \pi_1(X,p) \longrightarrow G \qquad \text{by} \qquad \tilde{\phi}\big([\alpha]\big) = \tilde{\Phi}(\alpha) \quad \forall\, \alpha \in \mathcal{L}(X,p).$$

Properties ($\tilde{\Phi}$1) and ($\tilde{\Phi}$2) insure that $\tilde{\phi}$ is well-defined (independent of the choice of representative α for $[\alpha]$) and is a group homomorphism, respectively. By ($\tilde{\Phi}$3),

$$\tilde{\phi}\big(j_m([\alpha]_{\mathcal{U}_m})\big) = \tilde{\phi}\big([\alpha]\big) = \tilde{\Phi}(\alpha) = \phi_m\big([\alpha]_{\mathcal{U}_m}\big) \qquad \forall\, [\alpha] \in \pi_1(\mathcal{U}_m, p),\ m=1,2,$$

i.e. $\phi_m = \tilde{\phi} \circ j_m$ as required.

First, we define

$$\Phi \colon \mathcal{L}(\mathcal{U}_1,p) \cup \mathcal{L}(\mathcal{U}_2,p) \longrightarrow G \qquad \text{by} \qquad \Phi(\alpha) = \begin{cases} \phi_1([\alpha]_{\mathcal{U}_1}), & \text{if } \alpha \in \mathcal{L}(\mathcal{U}_1,p); \\ \phi_2([\alpha]_{\mathcal{U}_2}), & \text{if } \alpha \in \mathcal{L}(\mathcal{U}_2,p). \end{cases}$$

Since $\phi_1 \circ i_1 = \phi_2 \circ i_2$, this map is well defined on the overlap:

$$\mathcal{L}(\mathcal{U}_1,p) \cap \mathcal{L}(\mathcal{U}_2,p) = \mathcal{L}(\mathcal{U}_1 \cap \mathcal{U}_2, p).$$

For if $\alpha \in \mathcal{L}(\mathcal{U}_1 \cap \mathcal{U}_2, p)$, then

$$\phi_1\big([\alpha]_{\mathcal{U}_1}\big) = \phi_1\big(i_1([\alpha]_{\mathcal{U}_1 \cap \mathcal{U}_2})\big) = \phi_2\big(i_2([\alpha]_{\mathcal{U}_1 \cap \mathcal{U}_2})\big) = \phi_2\big([\alpha]_{\mathcal{U}_2}\big).$$

Furthermore,

(Φ1) $[\alpha]_{\mathcal{U}_m} = [\beta]_{\mathcal{U}_m} \Longrightarrow \Phi(\alpha)=\Phi(\beta)$ for all $\alpha, \beta \in \mathcal{L}(\mathcal{U}_m, p)$, $m=1,2$;

($\Phi 2$) $\Phi(\alpha * \beta) = \Phi(\alpha)\Phi(\beta)$ for all $\alpha, \beta \in \mathcal{L}(\mathcal{U}_m, p)$, $m = 1, 2$;
($\Phi 3$) $\Phi(\alpha) = \phi_m\big([\alpha]_{\mathcal{U}_m}\big)$ for all $\alpha \in \mathcal{L}(\mathcal{U}_m, p)$, $m = 1, 2$.

Given $\alpha \in \mathcal{L}(X, 0)$, we define $\tilde{\Phi}(\alpha) \in H$ as follows. First,

$$\alpha \colon \big(I, \{0, 1\}\big) \longrightarrow (X, b)$$

is a continuous map. Take a subdivision of I into subintervals $[s_{k-1}, s_k]$, with $k = 1, \ldots, n$, such that the image of each subinterval under α is contained in $\mathcal{U}_m$ for $m = 1$ or 2 (depending on the subinterval). We do not require the subintervals to be of the same size. Define the path

$$\alpha_{[s_{k-1}, s_k]} \colon I \longrightarrow \mathcal{U}_m \subset X \qquad \text{by} \qquad \alpha_{[s_{k-1}, s_k]}(s) = \alpha\big(s_{k-1} + (s_k - s_{k-1})s\big).$$

This is just a reparametrization of the map $\alpha|_{[s_{k-1}, s_k]}$. For each $k = 1, \ldots, n-1$, choose a path

$$\gamma_k \colon (I, 0, 1) \longrightarrow \big(X, p, \alpha(s_k)\big) \qquad \text{s.t.} \qquad \operatorname{Im} \gamma_k \subset \mathcal{U}_m \quad \text{if} \quad \alpha(s_k) \in \mathcal{U}_m, \; m = 1, 2.$$

In particular, $\operatorname{Im} \gamma_k \subset \mathcal{U}_1 \cap \mathcal{U}_2$ if $\alpha(s_k) \subset \mathcal{U}_1 \cap \mathcal{U}_2$. Denote by γ_0 and γ_n the constant path at p. We then have

$$\alpha \qquad \gamma_0 * \alpha_{[s_0, s_1]} * \bar{\gamma}_1 * \gamma_1 * \alpha_{[s_1, s_2]} * \bar{\gamma}_2 * \ldots * \gamma_{n-1} * \alpha_{[s_{n-1}, s_n]} * \bar{\gamma}_n, \tag{1}$$

since $\bar{\gamma}_k * \gamma_k$ is path-homotopic to the constant path at $\alpha(s_k)$.

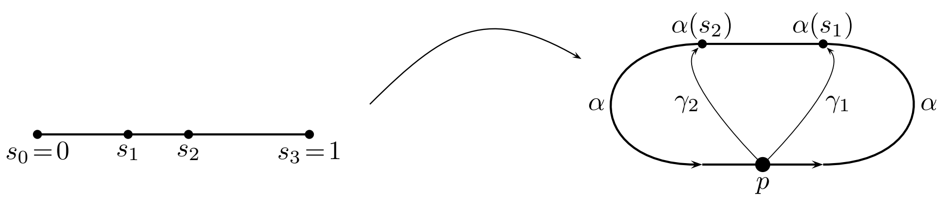


Figure 7: Splitting an Element of $\mathcal{L}(X, p)$ into Elements of $\mathcal{L}(\mathcal{U}_1, p)$ and $\mathcal{L}(\mathcal{U}_2, p)$

By construction, $\gamma_k * \alpha_{[s_{k-1}, s_k]} * \bar{\gamma}_k \in \mathcal{L}(\mathcal{U}_1, p) \cup \mathcal{L}(\mathcal{U}_1, p)$ for all $k = 1, \ldots, n$. Thus, by (1), $[\alpha] \in \pi_1(X, p)$ is in the subgroup generated by $\operatorname{Im} j_1$ and $\operatorname{Im} j_2$, i.e. this subgroup must be all of $\pi_1(X, p)$. We put

$$\tilde{\Phi}(\alpha) = \Phi\big(\gamma_0 * \alpha_{[s_0, s_1]} * \bar{\gamma}_1\big) * \Phi\big(\gamma_1 * \alpha_{[s_1, s_2]} * \bar{\gamma}_2\big) * \ldots * \Phi\big(\gamma_{n-1} * \alpha_{[s_{n-1}, s_n]} * \bar{\gamma}_n\big). \tag{2}$$

We claim that $\tilde{\Phi}(\alpha)$ is well-defined, i.e.
($I1$) $\tilde{\Phi}(\alpha)$ is independent of the choice of paths $\gamma_1, \ldots, \gamma_n$;
($I2$) $\tilde{\Phi}(\alpha)$ is independent of the choice of subdivision of I.
By ($I2$), ($\Phi 1$), and ($\Phi 3$), $\tilde{\Phi}$ satisfies ($\tilde{\Phi} 3$):

$$\tilde{\Phi}(\alpha) = \Phi\big(\gamma_0 * \alpha_{[0,1]} * \bar{\gamma}_1\big) = \Phi(\alpha) = \phi_m\big([\alpha]_{\mathcal{U}_m}\big) \qquad \forall\, \alpha \in \mathcal{L}(\mathcal{U}_m, p),\, m = 1, 2,$$

since in this case we can take $n = 1$. By (2), ($I2$), and ($\tilde{\Phi} 1$), $\tilde{\Phi}$ satisfies ($\tilde{\Phi} 2$), since we can simply use subdivisions for α and β to form an admissible subdivision for $\alpha * \beta$. Thus, it remains to verify ($I1$), ($I2$), and ($\tilde{\Phi} 1$).

We first show that replacing γ_k with another admissible path γ_k', for some $k = 1, \ldots, n-1$, does not change the product of the two terms in (2) that involve γ_k. Let

$$\beta_{k-1} = \gamma_{k-1} * \alpha_{[s_{k-1}, s_k]} \qquad \text{and} \qquad \beta_k = \alpha_{[s_k, s_{k+1}]} * \bar{\gamma}_{k+1}.$$

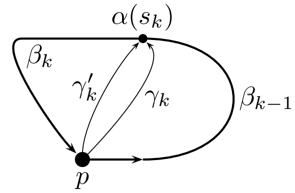


Figure 8: Verification of $(I1)$

If $\alpha(s_k)\in\mathcal{U}_m$, then $\gamma'_k*\bar{\gamma}_k, \gamma_k*\bar{\gamma}'_k\in\mathcal{L}(\mathcal{U}_m,p)$ and

$$\Phi\big(\gamma'_k*\bar{\gamma}_k\big)\Phi\big(\gamma_k*\bar{\gamma}'_k\big) = \Phi\big(\gamma'_k*\bar{\gamma}_k*\gamma_k*\bar{\gamma}'_k\big) = 1 \in G \tag{3}$$

by $(\Phi1)$ and $(\Phi2)$, since $\gamma'_k*\bar{\gamma}_k*\gamma_k*\bar{\gamma}'_k$ is path-homotopic in $\mathcal{U}_m$ to the constant path at x_0. Furthermore, if $\beta_{k-1}*\bar{\gamma}'_k\in\mathcal{L}(\mathcal{U}_m,p)$, then $\gamma'_k*\bar{\gamma}_k\in\mathcal{L}(\mathcal{U}_m,p)$ and

$$\Phi\big(\beta_{k-1}*\bar{\gamma}'_k\big)\Phi\big(\gamma'_k*\bar{\gamma}_k\big) = \Phi\big(\beta_{k-1}*\bar{\gamma}'_k*\gamma'_k*\bar{\gamma}_k\big) = \Phi\big(\beta_{k-1}*\bar{\gamma}_k\big) \tag{4}$$

by $(\Phi1)$ and $(\Phi2)$. By the same reasoning,

$$\Phi\big(\gamma_k*\bar{\gamma}'_k\big)\Phi\big(\gamma'_k*\beta_k\big) = \Phi\big(\gamma_k*\bar{\gamma}'_k*\gamma'_k*\beta_k\big) = \Phi\big(\gamma_k*\beta_k\big). \tag{5}$$

By (3)-(5),

$$\Phi\big(\beta_{k-1}*\bar{\gamma}'_k\big)\Phi\big(\gamma'_k*\beta_k\big) = \Phi\big(\beta_{k-1}*\bar{\gamma}'_k\big)\Phi\big(\gamma'_k*\bar{\gamma}_k\big)\Phi\big(\gamma_k*\bar{\gamma}'_k\big)\Phi\big(\gamma'_k*\beta_k\big) = \Phi\big(\beta_{k-1}*\bar{\gamma}_k\big)\Phi\big(\gamma_k*\beta_k\big),$$

i.e. the product of the two terms in (2) that involve γ_k does not change if γ_k is replaced by γ'_k.

In order to verify $(I2)$, it is sufficient to check that the product (2) does not change if one extra subdivision is added. Suppose $s'\in(s_{k-1},s_k)$ and γ' is a path from p to $\alpha(s')$ such that $\operatorname{Im}\gamma'\subset\mathcal{U}_m$ if $\alpha(s')\in\mathcal{U}_m$.

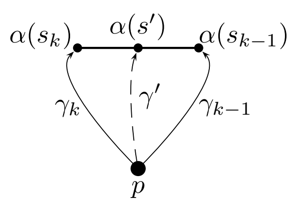


Figure 9: Verification of $(I2)$

Since the image of $[s_{k-1},s_k]$ under α is contained in $\mathcal{U}_m$ for some $m=1,2$,

$$\gamma_{k-1}*\alpha_{[s_{k-1},s']}*\bar{\gamma}', \gamma'*\alpha_{[s',s_k]}*\gamma_k \in \mathcal{L}(\mathcal{U}_m,p).$$

Thus, by $(\Phi1)$ and $(\Phi2)$,

$$\begin{aligned}\Phi\big(\gamma_{k-1}*\alpha_{[s_{k-1},s']}*\bar{\gamma}'\big)\Phi\big(\gamma'*\alpha_{[s',s_k]}*\gamma_k\big) &= \Phi\big(\gamma_{k-1}*\alpha_{[s_{k-1},s']}*\bar{\gamma}'*\gamma'*\alpha_{[s',s_k]}*\gamma_k\big)\\ &= \Phi\big(\gamma_{k-1}*\alpha_{[s_{k-1},s']}*\alpha_{[s',s_k]}*\gamma_k\big) = \Phi\big(\gamma_{k-1}*\alpha_{[s_{k-1},s_k]}*\gamma_k\big).\end{aligned}$$

In other words, the term in (2) involving $[s_{k-1},s_k]$ is the product of the terms involving its two subintervals in the new subdivision.

It remains to verify ($\tilde{\Phi}3$). Suppose $H: I\times I \longrightarrow X$ is a path homotopy between $\alpha, \beta \in \mathcal{L}(X,p)$. Choose subdivisions of the two components of $I\times I$ into subintervals $[s_{k-1}, s_k]$ and $[t_{l-1}, t_l]$ such that for every k, l

$$H\big([s_{k-1}, s_k]\times[t_{l-1}, t_l]\big) \subset \mathcal{U}_m$$

for some $m=1,2$, i.e. each of the small subrectangles of $I\times I$ is mapped by H either to $\mathcal{U}_1$ or $\mathcal{U}_2$. Let $\alpha^{(l)}\in\mathcal{L}(X,p)$ be the defined by

$$\alpha^{(l)}(s) = H(s, t_l).$$

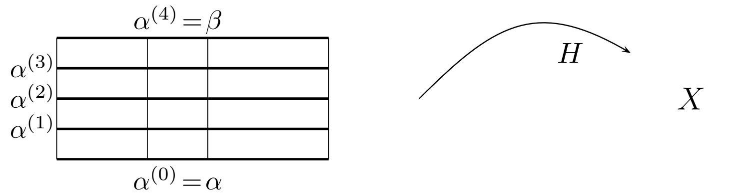


Figure 10: Splitting a Homotopy in X into Homotopies in $\mathcal{U}_1$ and $\mathcal{U}_2$

We will show that $\Phi(\alpha^{(l)})=\Phi(\alpha^{(l-1)})$ for all $l>0$. This implies that $\Phi(\alpha)=\Phi(\beta)$. Since each of the subrectangles $[s_{k-1}, s_k]\times[t_{l-1}, t_l]$ is mapped by H entirely into $\mathcal{U}_m$, for some $m=1,2$, we can subdivide I into the subintervals $[s_{k-1}, s_k]$ for the purposes of computing $\Phi(\alpha^{(l-1)})$ and $\Phi(\alpha^{(l)})$ via (2). We will show that the kth terms in the expressions in (2) for $\alpha^{(l-1)}$ and $\alpha^{(l)}$ are equal, for a compatible choice of paths γ_k connecting p to the "junction points" of $\alpha^{(l-1)}$ and $\alpha^{(l)}$. For each $k=1,\ldots,n$, let γ_k be a path from p to $\alpha^{(l-1)}(t_k)$ such that $\operatorname{Im}\gamma_k\subset\mathcal{U}_m$ if $\alpha^{(l-1)}(t_k)\in\mathcal{U}_m$. Let δ_k be the path from $\alpha^{(l-1)}(t_k)$ to $\alpha^{(l)}(t_k)$ corresponding to the vertical segment $s_k\times[t_{k-1}, t_k]$ in Figure 10, i.e.

$$\delta_k(s) = H\big(s_k, t_{k-1}+(t_k-t_{k-1})s\big).$$

Then, $\gamma_k'\equiv\gamma_k*\delta_k$ is a path from p to $\alpha^{(l)}(s_k)$ in $\mathcal{U}_m$.

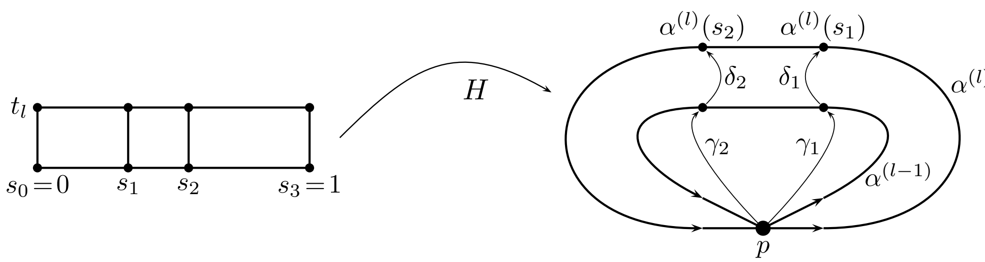


Figure 11: Verifying ($\tilde{\Phi}3$)

Furthermore, since H maps $[s_{k-1}, s_k]\times[t_{l-1}, t_l]$ into $\mathcal{U}_m$, the paths $\delta_{k-1}*\alpha^{(l)}|_{[s_{k-1},s_k]}*\bar{\delta}_k$ are path-homotopy in $\mathcal{U}_m$ (the homotopy is induced by $H|_{[s_{k-1},s_k]\times[t_{l-1},t_l]}$). Thus,

$$\begin{aligned}\big[\gamma_{k-1}'*\alpha^{(l)}|_{[s_{k-1},s_k]}*\bar{\gamma}_k'\big]_{\mathcal{U}_m} &= \big[\gamma_{k-1}*!\delta_{k-1}*\alpha^{(l)}|_{[s_{k-1},s_k]}*\bar{\delta}_k*\bar{\gamma}_k\big]_{\mathcal{U}_m} = \big[\gamma_{k-1}*\alpha^{(l-1)}|_{[s_{k-1},s_k]}*\bar{\gamma}_k\big]_{\mathcal{U}_m}\\ \Longrightarrow \qquad \Phi\big(\gamma_{k-1}'*\alpha^{(l)}|_{[s_{k-1},s_k]}*\bar{\gamma}_k'\big) &= \Phi\big(\gamma_{k-1}*\alpha^{(l-1)}|_{[s_{k-1},s_k]}*\bar{\gamma}_k\big)\end{aligned}$$

by ($\Phi1$), as claimed.