MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 1
Solution to Problem p100, 21

Suppose X is a topological space and A is a subset of X. Let $N(A)$ be the number of subsets of X that can be obtained from A by taking a sequence of complements and closures.
(a) Show that $N(A) \leq 14$.
(b) Give an example of a subset A of $\mathbb{R}$ such that $N(A) = 14$.

If A is any subset of X, let

$$f(A) = \bar{A} \qquad \text{and} \qquad g(A) = X - A.$$

Since $ff = f$ and $gg = \mathrm{Id}$, there are only two essentially different sequences of f and g to consider:

$$fgfgfg\ldots \qquad gfgfgf\ldots,$$

where gf means $gf(A) = f(g(A))$.

Lemma 1 *If A is a closed subset of X, then $N(A) \leq 6$.*

Proof: (1) Since $f(A) = A$, we only need to consider the sequence $gfgfgf\ldots$. Define

$$\mathring{A} = X - \overline{X-A} = gfg(A) \qquad \text{and} \qquad B = \overline{\mathring{A}} = gfgf(A).$$

Then $B \subset A$, since

$$B = \overline{X - \overline{X-A}} \subset \overline{X - (X-A)} = \bar{A} = A.$$

(2) We next show that $\mathring{A} = \mathrm{Int}\,A$. Since $\mathring{A}$ is an open set and $\mathring{A} \subset B \subset A$, $\mathring{A} \subset \mathrm{Int}\,A$. On the other hand, since $\mathrm{Int}\,A$ is an open set contained in A, $X - \mathrm{Int}\,A$ is a closed set containing $X - A$. Thus,

$$\overline{X-A} \subset X - \mathrm{Int}\,A \qquad \Longrightarrow \qquad \mathrm{Int}\,A \subset X - \overline{X-A} = \mathring{A}\,.$$

(3) We now show that $\mathrm{Int}\,B = \mathrm{Int}\,A$. Since $B \subset A$, $\mathrm{Int}\,B \subset \mathrm{Int}\,A$. On the other hand, $\mathrm{Int}\,A$ is an open set contained in B by (2) above and the definitions in (1).
(4) From (2) and (3), we obtain

$$X - \overline{X-B} = \mathrm{Int}\,B = \mathrm{Int}\,A \qquad \Longrightarrow \qquad X - \overline{X - \overline{X - \overline{X-A}}} = X - \overline{X-A}$$
$$\Longrightarrow \qquad \overline{X - \overline{X - \overline{X-A}}} = \overline{X-A}.$$

This gives a sequence of 6 possibly different subsets of X, after which we get a repetition:

$$A \longrightarrow X-A \longrightarrow \overline{X-A} \longrightarrow X - \overline{X-A} \longrightarrow \overline{X - \overline{X-A}} \longrightarrow X - \overline{X - \overline{X-A}}$$
$$\longrightarrow \overline{X - \overline{X - \overline{X-A}}} = \overline{X-A}.$$

Suppose now that A is an arbitrary subset of A. The two possible sequences of subsets starting with A begin with

$$A \longrightarrow \bar{A} \qquad \text{and} \qquad A \longrightarrow X-A \longrightarrow \overline{X-A}.$$

Since $\bar{A}$ is closed, by Lemma 1 there can at most 5 other distinct subsets of X arising from the first sequence. Similarly, there can at most 5 other distinct subsets arising from the second sequence. This implies (a).

Here is an example when $X=\mathbb{R}$ and $N(A)=14$.

$$\begin{array}{ll}
A = \big((0,1)-\mathbb{Q}\big)\cup(2,3)\cup(3,4)\cup\{5\} & X-A = (-\infty,0]\cup\big((0,1)\cap\mathbb{Q}\big)\cup[1,2]\cup\{3\}\cup[4,5)\cup(5,\infty) \\
\bar{A} = [0,1]\cup[2,4]\cup\{5\} & \overline{X-A} = (-\infty,2]\cup\{3\}\cup[4,\infty) \\
X-\bar{A} = (-\infty,0)\cup(1,2)\cup(4,5)\cup(5,\infty) & X-\overline{X-A} = (2,3)\cup(3,4) \\
\overline{X-\bar{A}} = (-\infty,0]\cup[1,2]\cup[4,\infty) & \overline{X-\overline{X-A}} = [2,4] \\
X-\overline{X-\bar{A}} = (0,1)\cup(2,4) & X-\overline{X-\overline{X-A}} = (-\infty,2)\cup(4,\infty) \\
\overline{X-\overline{X-\bar{A}}} = [0,1]\cup[2,4] & \overline{X-\overline{X-\overline{X-A}}} = (-\infty,2]\cup[4,\infty) \\
X-\overline{X-\overline{X-\bar{A}}} = (-\infty,0)\cup(1,2)\cup(4,\infty) & X-\overline{X-\overline{X-\overline{X-A}}} = (2,4).
\end{array}$$

In order to come up with an example as above, one could try to apply the proof of part (a) to various subsets of $\mathbb{R}$. First, one could look for a closed subset A of $\mathbb{R}$ such that $N(A)=6$. There are two types of "standard" closed subsets: a single-point set, e.g. $\{0\}$, and a closed interval, e.g. $[0,1]$. These two give us

$$\begin{array}{l}
\{0\} \longrightarrow(-\infty,0)\cup(0,\infty)\longrightarrow\mathbb{R} \longrightarrow\emptyset \longrightarrow\emptyset \longrightarrow\mathbb{R}\ldots \\
[0,1]\longrightarrow(-\infty,0)\cup(1,\infty)\longrightarrow(-\infty,0]\cup[1,\infty)\longrightarrow(0,1)\longrightarrow[0,1]\longrightarrow(-\infty,0)\cup(1,\infty)\ldots
\end{array}$$

In either case, we get 4 distinct subsets of $\mathbb{R}$, instead of 6. On the other hand, in the second case they cycle through the initial subset A, while in the first they do not. This suggests we should try a combination of subsets of the two "standard" types, e.g. $[0,1]\cup\{2\}$:

$$\begin{array}{l}
[0,1]\cup\{2\}\longrightarrow(-\infty,0)\cup(1,2)\cup(2,\infty)\longrightarrow(-\infty,0]\cup[1,\infty) \\
\qquad\longrightarrow(0,1) \longrightarrow[0,1] \longrightarrow(-\infty,0)\cup(1,\infty)\ldots
\end{array}$$

This indeed works. Of course, we can add in more points and intervals.

Based on the proof of part (a) in order to find a subset A of $\mathbb{R}$ such that $N(A)=14$ we should look at subsets A of $\mathbb{R}$ such that

$$\bar{A}\neq A \qquad \text{and} \qquad \overline{X-A}\neq X-A,$$

i.e. A is neither closed nor open. In addition, based on the previous paragraph, $\bar{A}$ and $\overline{X-A}$ should contain a closed interval and an isolated point, i.e. A and $X-A$ should contain “most” of an interval and an isolated point. If

$$\bar{A} = [0,1]\cup\{2\}$$

similarly to the previous paragraph, we could then try

$$A = (0,1/2)\cup(1/2,1)\cup\{2\} \qquad\Longrightarrow\qquad X-A = (-\infty,0]\cup\{1/2\}\cup[1,2)\cup(2,\infty).$$

The two possible sequences of subspaces of $\mathbb{R}$ obtained from A are

$$\begin{aligned}
&A = (0,1/2)\cup(1/2,1)\cup\{2\} && X-A = (-\infty,0]\cup\{1/2\}\cup[1,2)\cup(2,\infty)\\
&\bar{A} = [0,1]\cup\{2\} && \overline{X-A} = (-\infty,0]\cup\{1/2\}\cup[1,\infty)\\
&X-\bar{A} = (-\infty,0)\cup(1,2)\cup(2,\infty) && X-\overline{X-A} = (0,1/2)\cup(1/2,1)\\
&\overline{X-\bar{A}} = (-\infty,0]\cup[1,\infty) && \overline{X-\overline{X-A}} = [0,1]\\
&X-\overline{X-\bar{A}} = (0,1) && X-\overline{X-\overline{X-A}} = (-\infty,0)\cup(1,\infty)\\
&\overline{X-\overline{X-\bar{A}}} = [0,1] && \overline{X-\overline{X-\overline{X-A}}} = (-\infty,0]\cup[1,\infty)\\
&X-\overline{X-\overline{X-\bar{A}}} = (-\infty,0)\cup(1,\infty) && X-\overline{X-\overline{X-\overline{X-A}}} = (0,1)
\end{aligned}$$

In particular, starting with $\bar{A}$ and $\overline{X-\bar{A}}$, we obtain 6 distinct subsets, in each case. However, in order for the upper bound of 14 on the total number of subsets to be achieved, the two sequences must be disjoint; see the end of the proof of part (a).

The two sequences are not disjoint in this case. From the proof of Lemma 1, we know that the repeating parts of the two sequences are determined by the open sets $\mathrm{Int}\,\bar{A}$, or equivalently $X-\overline{\mathrm{Int}\,\bar{A}}$, and $\mathrm{Int}\,\overline{X-A}$. Thus, we need

$$X-\overline{\mathrm{Int}\,\bar{A}} \neq \mathrm{Int}\,\overline{X-A}.$$

By (2) of the proof of Lemma 1,

$$\mathrm{Int}\,\bar{A} = X-\overline{X-\bar{A}} \supset X-\overline{X-A} \qquad\Longrightarrow\qquad \overline{\mathrm{Int}\,\bar{A}} \supset X-\overline{X-A} \qquad\Longrightarrow\qquad X-\overline{\mathrm{Int}\,\bar{A}} \subset \overline{X-A}.$$

Since $X-\overline{\mathrm{Int}\,\bar{A}}$ is an open subset of X, the last inclusion implies that

$$X-\overline{\mathrm{Int}\,\bar{A}} \subset \mathrm{Int}\,\overline{X-A}.$$

Thus, we need to find a subset A of $\mathbb{R}$ such that

$$X-\overline{\mathrm{Int}\,\bar{A}} \subsetneq \mathrm{Int}\,\overline{X-A} \qquad\Longleftrightarrow\qquad \overline{\mathrm{Int}\,\bar{A}}\cap\mathrm{Int}\,\overline{X-A} \neq \emptyset \qquad\Longleftrightarrow\qquad \mathrm{Int}\,\bar{A}\cap\mathrm{Int}\,\overline{X-A} \neq \emptyset.$$

The last two conditions are equivalent because $\mathrm{Int}\,\overline{X-A}$ is an open set. Since $\mathrm{Int}\,\bar{A}\cap\mathrm{Int}\,\overline{X-A}$ is open, it must contain an interval (a,b) with $a<b$ if it is nonempty. Thus, we need to find A such that the closures of $(a,b)\cap A$ and $(a,b)-A$ are both $[a,b]$, for some $a<b$. This can be achieved by adding $(3,4)\cap\mathbb{Q}$ to the subset A of $\mathbb{R}$ described in the previous paragraph, i.e. the subset

$$A = (0,1/2)\cup(1/2,1)\cup\{2\}\cup\big((3,4)\cap\mathbb{Q}\big)$$

of $\mathbb{R}$ should provide the desired example. We know that it does, since this subset is analogous to the one used in the example for part (b).