Fundamental group fact sheet

Let X be a topological space. The set of homotopy classes of loops in X with the base-point $x_0 \in X$ is denoted by $\pi_1(X, x_0)$ and is called *the fundamental group of* X. The fundamental group is indeed a group. The group structure is given by the multiplication of loops (going around two loops successively) If X is path connected, then the fundamental groups with different base points are isomorphic. In this case, they are denoted simply by $\pi_1(X)$.

Example 1. The fundamental group of the circle is $\mathbb{Z}$. The isomorphism $\pi_1(S^1) \cong \mathbb{Z}$ is defined as follows. To any loop, we assign the number of times it goes around the circle.

Example 2. The fundamental group of $\mathbb{R}^n$ is trivial. Indeed, any loop in $\mathbb{R}^n$ can be contracted to 0 homotetically.

For any continuous map $f : X \to Y$ there is the corresponding homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$. The class $[\gamma]$ of a loop $\gamma : [0, 1] \to X$ gets mapped to the class of the loop $f \circ \gamma$ under this homomorphism.

If $f : X \to Y$ and $g : Y \to Z$ are continuous map, then $(g \circ f)_* = g_* \circ f_*$ on the level of the fundamental group. This statement has the following important corollary:

Proposition 0.1. *If topological spaces are homeomorphic, then their fundamental groups are isomorphic.*

Actually, the assumption of this proposition can be relaxed. Topological spaces X and Y are said to be *homotopy equivalent*, if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity self-map of Y, and $g \circ f$ is homotopic to the identity self-map of X.

Proposition 0.2. *If topological spaces are homotopy equivalent, then they have isomorphic fundamental groups.*

Consider a continuous map $p : T \to X$ of a path connected space T to a path connected space X. Suppose that any point in X has a neighborhood U such that $p^{-1}(U)$ is homeomporphic to the direct product $U \times Z$ of U with some discrete space Z. Suppose also that this homeomorphism composed with the standard projection of $U \times Z$ to the first factor, gives the identity. Then p is called a *covering*, T is called the *covering space* and X is called the *base space* of the covering p.

If we drop the condition of T being path connected in the definition of a covering, then we obtain the definition of a covering in a wider sense. We will call these coverings in a wider sense "coverings" (in quotation marks).

Proposition 0.3. *Any "covering" over* $[0, 1]$ *is homeomorphic to a direct product of* $[0, 1]$ *with a discrete space. This homeomorphism composed with the standard projection of this direct product to* $[0, 1]$*, gives the identity map.*

Corollary 0.4. *Let* $p : T \to [0, 1]$ *be a "covering". Consider a path* $\gamma : [0, 1] \to [0, 1]$ *in* $[0, 1]$*. For any point* $\tilde{x}_0 \in T$*, there exists a unique path* $\tilde{\gamma}$ *in* T *such that* $p \circ \tilde{\gamma} = \gamma$ *and* $\tilde{\gamma}(0) = \tilde{x}_0$.

Given a covering $p : T \to X$ and a continuous map $f : Y \to X$, we can define a covering over Y, which is called the *covering on* Y *induced by the map* f. Namely,

$$T' = \{(y, z) \in Y \times T |\ p(z) = f(y)\}$$

The covering $p' : T' \to Y$ is defined by the formula $p'(y, z) = y$.

From the existence of induced coverings and the Corollary stated above, it follows that

Theorem 0.5 (path lifting)**.** *Consider a "covering" $p : T \to X$ and a path $\gamma : [0, 1] \to X$. For any point $\tilde{x}_0 \in T$, there exists a unique path $\tilde{\gamma}$ in T such that $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$.*

Theorem 0.6 (homotopy lifting)**.** *Consider a homotopy f_t of a map f_0 of a topological space Y to the base space X of a "covering" $p : T \to X$. Suppose that f_0 lifts to a map $F_0 : Y \to T$ so that $p \circ F_0 = f_0$. Then there is a homotopy F_t of the map F_0 such that $p \circ F_t = f_t$.*

The fundamental theorem stated above imply the following important corollaries:

Corollary 0.7. *Consider a covering $p : T \to X$. The induced map $p_* : \pi_1(T) \to \pi_1(X)$ is injective.*

Corollary 0.8. *The set $\pi_1(X)/p_*\pi_1(T)$ is in a 1-1 correspondence with the preimage $p^{-1}(x)$ of any point $x \in X$.*

Example 3. From Corollary 0.8 it follows that the fundamental group of $\mathbb{R}P^n$ is $\mathbb{Z}/2\mathbb{Z}$ (this is the only group with 2 elements).

Example 4. We can extend Corollary 0.8 to a proof of the fact that $\pi_1(8)$ is the free group with 2 generators. Here 8 is the figure eight. We can define a covering T of 8 as the infinite graph such that its edges are marked with labels a, b, a^{-1} and b^{-1} and such that each vertex is incidend to exactly 4 edges, one of each kind. The space T is *simply connected* (i.e. it is path connected, and its fundamental group is trivial). The projection $p : T \to X$ is defined as follows. All vertices go to the vertex of 8 (the point where the two circles touch). A path along an a-edge corresponds to a path along the upper circle of 8 in the counterclockwise direction. A path along an a^{-1}-edge corresponds to a path along the upper circle of 8 in the clockwise direction. A path along a b-edge corresponds to a path along the lower circle of 8 in the counterclockwise direction. A path along a b^{-1}-edge corresponds to a path along the lower circle of 8 in the clockwise direction.

Let us state some applications of the fundamental group.

No retraction theorem. A subspace A of a topological space X is called a *retract* of X if there is a continuous map $f : X \to A$ such that f restrict to the identity map on A.

Theorem 0.9. *The boudnary circle S^1 of a disk D^2 is not a retract of the disk.*

Proof. The identity self-map of S^1 induces the identity self-map of the fundamental group. If there were a retraction $f : D^2 \to S^1$, then the identity self-map of S^1 would factor through the inclusion $S^1 \to D^2$, which is nulhomotopic. A contadiction. □

Theorem 0.9 generalizes to higher dimensions:

Theorem 0.10. *The boudnary sphere S^n of a disk D^{n+1} is not a retract of the disk.*

Fundamental theorem of algebra. Using the fundamental group, we can prove that any polynomial with constant coefficients has at least one complex root, which is the most difficult part of the fundamental theorem of algebra. Let $f : \mathbb{C} \to \mathbb{C}$

be a polynomial. Restrict it to a very big circle S around 0. We can prove that $f(z) = z^n + g(z)$, where n is the degree of f and $|g(z)| < |z^n|$ on S. Then f restricts to a map of S to $\mathbb{C} - 0$, which is clearly homotopic to z^n. This map induces a nontrivial homomorphism of $\pi_1(S)$ to $\pi_1(\mathbb{C} - 0)$, namely, the multiplication by n.

Therefore, f cannot map $\mathbb{C}$ to $\mathbb{C} - 0$, otherwise its restriction to S would be null-homotopic. It follows that f attains the value 0 somewhere.

Antipode-preserving maps. A map $f : S^n \to S^n$ is called *antipode-preserving* if for any $x \in S^n$ we have $f(-x) = -f(x)$.

Theorem 0.11. *No antipose-preserving map of S^n to S^n is nulhomotopic.*

Indeed, any antipode-preserving map $f : S^n \to S^n$ gives rise to a self-map of $\mathbb{R}P^n$. The latter induces the identity homomorphism on the level of the fundamental group.

Corollary 0.12. *There is no antipode-preserving map $f : S^{n+1} \to S^n$.*

Corollary 0.13 (Borsuk–Ulam theorem)**.** *For any continuous map $f : S^n \to \mathbb{R}^n$, there is a point $x \in S^n$ such that $f(x) = f(-x)$.*

Ham sandwich theorem. Consider a sandwitch consisting of bread, butter and meat. It is always possible to cut this sandwich by a plane so that bread, butter and meat are divided equally, i.e. there are equal volumes of them in both half-spaces.

This statement generalizes to the following theorem:

Theorem 0.14. *Suppose that $\mu_1, \dots, \mu_n$ are n measures on $\mathbb{R}^n$ that are absoltely continuous with respect to the Lebesgue measure (i.e. the μ_i-volume of any set of Lebesgue measure zero is also zero for $i = 1 \dots n$). Then there exists a hyperplane H in $\mathbb{R}^n$ dividing $\mathbb{R}^n$ into two half-spaces H_+ and H_- and such that*

$$\int_{H_+} d\mu_i = \int_{H_-} d\mu_i, \quad i = 1 \dots n.$$

Proof. Consider the map f of the set of all half-spaces H_+ to $\mathbb{R}^n$ such that H_+ is mapped to the vector

$$\left(\int_{H_+} d\mu_1, \dots, \int_{H_+} d\mu_n\right) \in \mathbb{R}^n.$$

The source space can be embedded into S^n and f extends to S^n. It remains to use the Borsuk–Ulam theorem. □