Axiom of Choice. The *Axiom of Choice* states that for any collection $\mathcal{S}$ of sets there exists a *choice function* $\mathcal{S} \to \bigcup \mathcal{S}$ such that $\sigma(X) \in X$ for all $X \in \mathcal{S}$. We will derive the Zorn lemma directly from the axiom of choice.

Theorem 1 (Zorn)**.** *Let X be a partially ordered set. Suppose that for any linearly ordered subset A of X there is an upper bound, i.e. an element $\hat{a} \in X$ such that $a \leq \hat{a}$ for all $a \in A$. Then X has a maximal element, i.e. an element $\hat{x} \in X$ such that the inequality $x \geq \hat{x}$ implies $x = \hat{x}$.*

Proof. The proof goes in several steps.

1) Fix any element $x_0 \in X$.

2) Suppose that X has no maximal elements. Then for any linearly ordered subset $A \subset X$, the set of all upper bounds not belonging to A, is nonempty. Choose one upper bound $\sigma(A)$ from this set. Thus we have $A \leq \sigma(A)$ (i.e., $a \leq \sigma(A)$ for all $a \in A$) and $a \notin A$. Let us fix this choice function σ for the rest of the proof.

3) Define a *chain* in X to be a subset $C \subset X$ satisfying the following properties:

- We have $x_0 \in C$.
- The subset C is well-ordered.
- For any $x \in C$, we have $\sigma\{y \in C |\ y < x\} = x$

4) Chains exist, e.g. $\{x_0\}$ is a chain.

5) If C is a chain and $x \in C$, then $C_x = \{y \in C |\ y < x\}$ is also a chain (this is obvious). By definition of chains, we have $\sigma(C_x) = x$.

6) Suppose that $C' \subset C$ are chains. If $C' \neq C$, then there is an element $x \in C$ such that $C' = C_x$. To prove this, we first define x to be the mininal (smallest) element of the set $C - C'$. Then it is clear that $C_x \subset C'$. Let x' be the minimal element of $C' - C_x$, so that we have $C'_{x'} \subset C_x$. The element x' cannot be less than x, otherwise it would be in C_x. Hence, $x' \geq x$. It follows that $C'_{x'} \supset C_x$. We see that $C_x = C'_{x'}$ as we have proved both inclusions. But now $\sigma(C_x) = x$ and $\sigma(C'_{x'}) = x'$, therefore, $x' = x$. But x' belongs to C' whereas x does not. Contradiction.

7) For any pair of chains C and C', we have either $C \subset C'$ or $C' \subset C$. Indeed, consider the chain $C \cap C'$. By step 6), if it does not coincide with either C or C', then it has the form $C_x = C'_{x'}$ for some $x \in C$ and $x' \in C'$. In particular, the elements x and x' do not belong to the intersection $C \cap C'$, while x belongs to C and x' belongs to C'. But now $x = \sigma(C_x) = \sigma(C'_{x'}) = x'$. Contradiction.

8) Take the union of $\hat{C}$ all chains. We claim that $\hat{C}$ is also a chain. It is clear from steps 6) and 7) that $\hat{C}$ is linearly ordered and that $\sigma(\hat{C}_x) = x$ for all $x \in C$. It only remains to prove that $\hat{C}$ is well-ordered. [This is an exercise from Homework 4].

9) The chain $\hat{C}$ is the largest chain. Now consider $\hat{x} = \sigma(\hat{C})$. Clearly, $\hat{C} \cup \{\hat{x}\}$ is a chain that is bigger than $\hat{C}$. Contradiction, which proves the Zorn lemma. $\square$

Theorem 2 (Zermelo)**.** *Any set admits a well-ordering.*

Proof. This theorem can be proved in almost the same way as the Zorn lemma. Namely, we need to fix a point $x_0 \in X$, and a choice function σ that maps any subset $A \subset X$ to a point $\sigma(A) \notin A$.

Define a *chain* in X to be a subset $C \subset X$ together with a well-ordering of C such that x_0 is the minimal element of C, and $\sigma\{y \in C |\ y < x\} = x$ for any $x \in C$.

We can repeat steps 4)-9) to show that the union $\hat{C}$ of all chains is well-ordered. The chain $\hat{C}$ is the largest, i.e. it contains all other chains. To get the contradiction, we now can define an order on $\hat{C} \cup \{\sigma(\hat{C})\}$ such that this will be a chain. It suffices to declare $\sigma(\hat{C})$ to be the largest element. Contradiction with the fact that $\hat{C}$ is the largest chain. $\square$