Intersections on Grassmannians of Two-Planes
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1 Flags and Schubert cycles

A flag $\mathbf{V}$ in $\mathbb{C}^n$ is an increasing sequence of $n+1$ linear subspaces of $\mathbb{C}^n$:

$$\mathbf{V} = (V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq \ldots V_{n-1} \subsetneq V_n). \tag{1}$$

Thus, V_i is a linear subspace of $\mathbb{C}^n$ of dimension i containing V_{i-1}; in particular, $V_0 = \{0\}$ and $V_n = \mathbb{C}^n$. The standard flag $\mathbf{V}^{\mathrm{std}}$ on $\mathbb{C}^n$ is the sequence as above with $V_i^{\mathrm{std}} = \mathbb{C}^i \times \{0\}^{n-i} \equiv \mathbb{C}^i$.

Let $G(2,n)$ denote the Grassmannian of two-dimensional linear subspaces of $\mathbb{C}^n$. For any non-negative integers a, b and a flag $\mathbf{V}$ as in (1), let

$$\sigma_{ab}(\mathbf{V}) = \big\{ P \in G(2,n) \colon P \subset V_{n-b}, \; P \cap V_{n-1-a} \neq \{0\} \big\}. \tag{2}$$

The second condition implies that $n-1-a \geq 1$ if $\sigma_{ab}(\mathbf{V}) \neq \emptyset$, so that $n-2 \geq a$. Since P is a linear subspace of $\mathbb{C}^n$ of dimension 2 and V_{n-1-b} is a linear subspace of V_{n-b}, the first condition in (2) and linear algebra imply that for any $P \in \sigma_{ab}(\mathbf{V})$

$$\dim P \cap V_{n-1-b} \geq \dim P + \dim V_{n-1-b} - \dim V_{n-b} = 1 \qquad \Longrightarrow \qquad P \cap V_{n-1-b} \neq \{0\}.$$

Thus, if $n-1-a > n-1-b$, the second condition in (2) is meaningless since then $V_{n-1-b} \subset V_{n-1-a}$ and $P \cap V_{n-1-a} \neq \{0\}$. Therefore, one always requires that $a \geq b$. Similarly, since V_{n-b} is defined only for $b \geq 0$, we assume that $b \geq 0$. In summary, $\sigma_{ab}(\mathbf{V}) \subset G(2,n)$ is defined by (2) whenever $n-2 \geq a \geq b \geq 0$; otherwise, it is defined to be empty.

If $P \in G(2,n)$, then $P \subset V_{n-0}$ and by linear algebra

$$\dim P \cap V_{n-1} \geq \dim P + \dim V_{n-1} - \dim V_n = 1 \qquad \Longrightarrow \qquad P \cap V_{n-1-0} \neq \emptyset.$$

Thus, $\sigma_{00}(\mathbf{V}) = G(2,n)$. In general, the integers a and b in (2) indicate how much earlier a typical element P of $\sigma_{ab}(\mathbf{V})$ satisfies the containment and intersection conditions with respect to the given flag. By (2),

$$\sigma_{ab}(\mathbf{V}) \supset \sigma_{a'b'}(\mathbf{V}) \qquad \text{if} \quad a \leq a', \; b \leq b', \; (a,b) \neq (a',b'). \tag{3}$$

The subspace $\sigma_{ab}(\mathbf{V})$ is a compact analytic subvariety of the complex manifold $G(2,n)$ of (complex) codimension $a+b$. The complement $\sigma_{ab}^*(\mathbf{V})$ in $\sigma_{ab}(\mathbf{V})$ of the finitely subspaces $\sigma_{a'b'}(\mathbf{V})$ with (a', b') as in (3) is a smooth complex submanifold of $G(2,n)$, but is generally not compact. Since the dimension of $G(2,n)$ is $2(n-2)$, this means that $\sigma_{ab}(\mathbf{V})$ can be written as a disjoint union of finitely many complex manifolds with the largest dimension equal $2(n-2) - (a+b)$.

If $\mathbf{V}$ and $\mathbf{V}'$ are two different flags in $\mathbb{C}^n$, there is a smooth path of flags $\mathbf{V}^{(t)}$, with $t \in [0,1]$, so that $\mathbf{V}^{(0)} = \mathbf{V}$ and $\mathbf{V}^{(1)} = \mathbf{V}'$. The smooth (generally non-compact) manifold with boundary

$$M = \big\{ (t, P) \in [0,1] \times G(2,n) \colon P \in \sigma_{ab}^*(\mathbf{V}^{(t)}) \big\}$$

is then a pseudocycle equivalence (even a homotopy) between the cycles $\sigma_{ab}(\mathbf{V})$ and $\sigma_{ab}(\mathbf{V}')$. Thus, the equivalence class of $\sigma_{ab}(\mathbf{V})$ as a cycle in $G(2,n)$ is independent of the flag $\mathbf{V}$ and is denoted σ_{ab}. It is customary to abbreviate σ_{a0} as σ_a.

2 Intersections of cycles

Given equivalence classes σ_{ab} and $\sigma_{a'b'}$, their intersection product $\sigma_{ab} \cdot \sigma_{a'b'}$ is the equivalence class of the cycle $\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}')$ for a generic pair of flags $\mathbf{V}$ and $\mathbf{V}'$ on $\mathbb{C}^n$. Similarly to the previous paragraph, any two pairs of such flags are homotopic, so that the equivalence class of the cycle $\sigma_{ab}(\mathbf{V}) \cap \sigma_{a'b'}(\mathbf{V}')$ is independent of the generic pair $(\mathbf{V}, \mathbf{V}')$. The codimension of the cycle $\sigma_{ab} \cdot \sigma_{a'b'}$ is given by

$$\operatorname{codim} \sigma_{ab} \cdot \sigma_{a'b'} = \operatorname{codim} \sigma_{ab} + \operatorname{codim} \sigma_{a'b'} = (a+b) + (a'+b').$$

If $\sigma_{a_1 b_1}, \ldots, \sigma_{a_k b_k}$ are k cycles, the codimension of the cycle $\sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}$ is thus

$$\operatorname{codim}\left(\sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}\right) = \sum_{i=1}^{i=k} (a_i + b_i).$$

If this number equals $2(n-2)$, then the dimension of this cycle in $G(2, n)$ is

$$\dim\left(\sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}\right) = \dim G(2, n) - \operatorname{codim}\left(\sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}\right) = 0,$$

i.e. $\sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}$ is simply a finite collection of points. The number of these points is denoted by

$$\left\langle \sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}, G(2, n) \right\rangle \in \mathbb{Z}.$$

These numbers are called intersection numbers of Schubert cycles.

The above intersection numbers satisfy the following identities:

$$\left\langle \sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}, G(2, n) \right\rangle = \left\langle \sigma_{a_1 - b_1} \cdot \ldots \cdot \sigma_{a_k - b_k}, G(2, n - b_1 - \ldots - b_k) \right\rangle; \tag{S1}$$

$$\left\langle \sigma_{n-2} \cdot \sigma_{n-2}, G(2, n) \right\rangle = 1; \tag{S2}$$

$$\left\langle \sigma_{a_1} \sigma_{a_2} \sigma_{a_3}, G(2, n) \right\rangle = 1 \quad \text{if} \quad n - 2 \geq a_1, a_2, a_3 \geq 0, \ a_1 + a_2 + a_3 = 2n - 4; \tag{S3}$$

$$\sigma_{a_1} \cdot \sigma_{a_2} = \sum_{c \geq a_1, a_2} \sigma_{c, a_1 + a_2 - c}. \tag{S4}$$

It is straightforward to verify the first three identities above directly directly from the definitions. We verified (S1)-(S3) directly from the definition during one of the discussion sessions. The second identity is the $a_1, a_2 = n-2$, $a_3 = 0$ case of (S3).

The relation (S4) is known as Pieri's formula. It is actually an immediate consequence of (S1), (S3), and the structure of the cohomology of $G(2, n)$. The latter implies that two cycles A and B in $G(2, n)$ are equivalent if and only if

$$\left\langle A \cdot \sigma_{de}, G(2, n) \right\rangle = \left\langle B \cdot \sigma_{de}, G(2, n) \right\rangle \qquad \forall d, e \in \mathbb{Z}.$$

Thus, in order to verify (S4), it is sufficient to show that

$$\left\langle \sigma_{a_1} \cdot \sigma_{a_2} \cdot \sigma_{de}, G(2, n) \right\rangle = \sum_{c \geq a_1, a_2} \left\langle \sigma_{c, a_1 + a_2 - c} \cdot \sigma_{de}, G(2, n) \right\rangle \qquad \forall d, e \in \mathbb{Z}. \tag{4}$$

We can assume that $n-2 \geq a_1, a_2 \geq 0$, $a_1 + a_2 + d + e = 2(n-2)$, and $n-2 \geq d \geq e \geq 0$; otherwise, both sides of (4) are zero. By (S1) and then (S3),

$$\big\langle \sigma_{a_1} \cdot \sigma_{a_2} \cdot \sigma_{de}, G(2,n) \big\rangle = \big\langle \sigma_{a_1} \cdot \sigma_{a_2} \cdot \sigma_{d-e}, G(2, n-e) \big\rangle$$
$$= \begin{cases} 1, & \text{if } \min(n-2, 2(n-2)-(a_1+a_2)) \geq d \geq (n-2) - \min(a_1, a_2), \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

with the above assumptions on d and e. Similarly, if $\min(n-2, a_1+a_2) \geq c \geq \max(a_1, a_2)$,

$$\big\langle \sigma_{c, a_1+a_2-c} \cdot \sigma_{de}, G(2,n) \big\rangle = \big\langle \sigma_{2c-a_1-a_2} \cdot \sigma_{d-e}, G(2, n+c-a_1-a_2-e) \big\rangle$$
$$= \begin{cases} 1, & \text{if } d = n-2+c-a_1-a_2; \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$\sum_{c \geq a_1, a_2} \big\langle \sigma_{c, a_1+a_2-c} \cdot \sigma_{de}, G(2,n) \big\rangle$$
$$= \begin{cases} 1, & \text{if } \min(n-2, 2(n-2)-(a_1+a_2)) \geq d \geq (n-2) - \min(a_1, a_2); \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Comparing (5) with (6), we obtain (4) and thus (S4). Unfortunately, this argument requires deep facts about cycles in $G(2,n)$. We give a direct argument below.

3   The intersection $\sigma_1 \cdot \sigma_1$ in $G(2,4)$

A special case of the direct argument is used in the book to show that

$$\sigma_1 \cdot \sigma_1 = \sigma_2 + \sigma_{11} \tag{7}$$

in $G(2,4)$. The argument in the book views the elements of $G(2,4)$ as lines in $\mathbb{P}^3$ by taking their projectivizations. Here is the argument by considering them as 2-planes in $\mathbb{C}^4$. We need to intersect

$$\sigma_1(\mathbf{V}^{\mathrm{std}}) \equiv \big\{ P \in G(2,4) \colon P \cap \mathbb{C}^2 \neq \{0\} \big\},$$
$$\sigma_1(\mathbf{V}) \equiv \big\{ P \in G(2,4) \colon P \cap V_2 \neq \{0\} \big\},$$

for a generic flag $\mathbf{V}$. For such a flag $\mathbb{C}^2 \cap V_2 = \{0\}$. However, we can move $\mathbf{V}$ so that $\mathbb{C}^2 \cap V_2 = \mathbb{C}$ and thus $\mathbb{C}^2 + V_2$ is a three-dimensional linear subspace of $\mathbb{C}^4$, which we can assume to be $\mathbb{C}^3$; it is spanned by the one-dimensional linear subspace $L_0 = \mathbb{C}$, a one-dimensional linear subspace L_1 in $\mathbb{C}^2$ different from L_0, and a one-dimensional linear subspace L_2 in V_2 different from L_0. An element of
$$\sigma_1(\mathbf{V}^{\mathrm{std}}) \cap \sigma_1(\mathbf{V}) \equiv \big\{ P \in G(2,4) \colon P \cap \mathbb{C}^2 \neq \{0\}, \ P \cap V_2 \neq \{0\} \big\}$$
must either contain the line $L_0 = \mathbb{C}$ common to $\mathbb{C}^2$ and V_2 or intersect $\mathbb{C}^2$ and $\mathbf{V}_2$ along some one-dimensional linear subspace $L_1 \subset \mathbb{C}^2$ and $L_2 \subset V_2$. In the latter case, P must lie in $\mathbb{C}^2 + V_2$.

Conversely, if P is a two-dimensional linear subspace of $\mathbb{C}^2 + V_2 = \mathbb{C}^3$, then P must intersect $\mathbb{C}^2$ and V_2 at least in a one-dimensional linear subspace, since by linear algebra

$$\dim P \cap \mathbb{C}^2 \geq \dim P + \dim \mathbb{C}^2 - \dim \mathbb{C}^3 \geq 1;$$
$$\dim P \cap V_2 \geq \dim P + \dim V_2 - \dim \mathbb{C}^3 \geq 1.$$

From this, we obtain

$$\sigma_1(\mathbf{V}^{\mathrm{std}}) \cap \sigma_1(\mathbf{V}) = \big\{ P \in G(2,n) \colon P \cap \mathbb{C}^{4-1-2} \neq \{0\} \big\}$$
$$\cup \big\{ P \in G(2,n) \colon P \subset \mathbb{C}^{4-1}, \ P \cap \mathbb{C}^{4-1-1} \neq \{0\} \big\}$$
$$= \sigma_2(\mathbf{V}^{\mathrm{std}}) \cup \sigma_{11}(\mathbf{V}^{\mathrm{std}}).$$

This implies (7) in the case of $G(2,4)$.

4 Proof of (S4) for $G(2,n)$

A similar argument extends to the general case of (S4) with some care. The formula (S4) holds for $G(2,2)$, with the only choices for a_1, a_2 being $a_1, a_2 = 0$; otherwise, both sides of (S4) vanish by definition. Suppose $n \geq 3$ and the formula holds for all $G(2,m)$ with $m < n$. We need to determine the intersection of

$$\sigma_a(\mathbf{V}^{\mathrm{std}}) \equiv \big\{ P \in G(2,n) \colon P \cap \mathbb{C}^{n-1-a} \neq \{0\} \big\},$$
$$\sigma_b(\mathbf{V}) \equiv \big\{ P \in G(2,n) \colon P \cap V_{n-1-b} \neq \{0\} \big\},$$

for a generic flag $\mathbf{V}$ as in (1).

Case 1: $a+b > n-2$. In this case,

$$\dim(\mathbb{C}^{n-1-a} + V_{n-1-b}) \leq \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} = n + (n-2) - (a+b) < n.$$

Thus, $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \{0\}$ if V_{n-1-b} is generic. We can also assume that $\mathbb{C}^{n-1-a} + \mathbf{V}_{n-1-b} \subset \mathbb{C}^{n-1}$. Then,

$$\sigma_a(\mathbf{V}^{\mathrm{std}}) \cap \sigma_b(\mathbf{V}) \equiv \big\{ P \in G(2,n) \colon P \cap \mathbb{C}^{n-1-a} \neq \{0\}, \ P \cap V_{n-1-b} \neq \{0\} \big\}$$
$$= \big\{ P \in G(2,n-1) \colon P \cap \mathbb{C}^{(n-1)-1-(a-1)} \neq \{0\}, \ P \cap V_{(n-1)-1-(b-1)} \neq \{0\} \big\}.$$

The last set is precisely the intersection of $\sigma_{a-1}(\mathbf{V}^{\mathrm{std}})$ with $\sigma_{b-1}(\mathbf{V})$ in $G(2,n-1)$.

By the inductive assumption,

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1} \quad \text{in} \ \ G(2,n-1).$$

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Thus, as a cycle in $G(2, n-1) \subset G(2, n)$, this intersection is equivalent to

$$\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \left\{ P \in G(2, n-1) \colon P \subset \mathbb{C}^{(n-1)-(a+b-c-1)}, \ P \cap C^{(n-1)-1-(c-1)} \neq \{0\} \right\}$$

$$= \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \left\{ P \in G(2, n) \colon P \subset \mathbb{C}^{n-(a+b-c)}, \ P \cap \mathbb{C}^{n-1-c} \neq \{0\} \right\} \tag{8}$$

$$= \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c,a+b-c}(\mathbf{V}_{\mathrm{std}}) = \sum_{c \geq a_1, a_2} \sigma_{c, a_1+a_2-c}.$$

The last equality above holds because $a_1 + a_2$ $c \leq n-24$ and $a_1 + a_2 > n-2$ This gives (S4).

Case 2: $a+b \leq n-2$. If $\mathbf{V}$ is a generic flag,

$$\dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) = \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} - \dim \mathbb{C}^n = n - 2 - (a+b) \geq 0.$$

However, we can move $\mathbf{V}$ so that $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \mathbb{C}^{n-1-(a+b)}$, and thus by linear algebra

$$\begin{aligned}
\dim(\mathbb{C}^{n-1-a} + V_{n-1-b}) &= \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} - \dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) \\
&= (n-1-a) + (n-1-b) - (n-1-(a+b)) \\
&= n-1.
\end{aligned}$$

Thus, we can assume that $\mathbb{C}^{n-1-a} + V_{n-1-b} = \mathbb{C}^{n-1}$.

Under the above assumption, the set

$$\sigma_a(\mathbf{V}^{\mathrm{std}}) \cap \sigma_b(\mathbf{V}) \equiv \left\{ P \in G(2, n) \colon P \cap \mathbb{C}^{n-1-a} \neq \{0\}, \ P \cap V_{n-1-b} \neq \{0\} \right\}$$

consists of the elements $P \in G(2, n)$ such that P either

- has a one-dimensional linear subspace in common with $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \mathbb{C}^{n-1-(a+b)}$ or

- intersects $\mathbb{C}^{n-1-a}$ and V_{n-1-b} along some one-dimensional linear subspaces $L_1 \subset \mathbb{C}^{n-1-a}$ and $L_2 \subset V_{n-1-b}$.

Since P is a two-dimensional linear subspace of $\mathbb{C}^n$, it must lie in $\mathbb{C}^{n-1-a} + V_{n-1-b} = \mathbb{C}^{n-1}$ in the latter case. From this, we obtain

$$\begin{aligned}
\sigma_a(\mathbf{V}^{\mathrm{std}}) \cap \sigma_b(\mathbf{V}) = &\left\{ P \in G(2, n) \colon P \cap \mathbb{C}^{n-1-(a+b)} \neq \{0\} \right\} \\
&\cup \left\{ P \in G(2, n) \colon P \subset \mathbb{C}^{n-1}, \ P \cap \mathbb{C}^{n-1-a} \neq \{0\}, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. P \cap V_{n-1-b} \neq \{0\} \right\}.
\end{aligned} \tag{9}$$

The first set on the right-hand side of (9) is precisely the cycle $\sigma_{a+b}(\mathbf{V}^{\mathrm{std}})$ in $G(2, n)$; this set is empty if $a+b > n-2$. The second set is the intersection of the cycles

$$\begin{aligned}
\sigma_{a-1}(\mathbf{V}^{\mathrm{std}}) &\equiv \left\{ P \in G(2, n-1) \colon P \cap \mathbb{C}^{(n-1)-1-(a-1)} \neq \{0\} \right\}, \\
\sigma_{b-1}(\mathbf{V}) &\equiv \left\{ P \in G(2, n-1) \colon P \cap V_{(n-1)-1-(b-1)} \neq \{0\} \right\},
\end{aligned}$$

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in $G(2, n-1)$. By the inductive assumption,

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1} \quad \text{in } G(2, n-1).$$

By (8), the second set in (9) as a cycle in $G(2, n-1) \subset G(2, n)$, is equivalent to

$$\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \{P \in G(2, n-1) : P \subset \mathbb{C}^{(n-1)-(a+b-c-1)}, \; P \cap C^{(n-1)-1-(c-1)} \neq \{0\}\}$$

$$= \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c,a+b-c}(\mathbf{V}_{\text{std}}).$$

(10)

Combining (9) and (10), we obtain (S4).