MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2018

Homework Assignment V

Written Assignment due on Tuesday, 11/20, at 1pm, in ESS 181

Please write up solutions to all of the following:

- Chapter 6, #6, 7 or Problem K below (one of the three);
- Problem J below, Chapter 7, #2,6 (all three);
- Chapter 7, #5,9, or 10 (one of the three).

Problem J

Recall that an element ℓ of $\mathbb{P}^n$ is a complex one-dimensional linear subspace of $\mathbb{C}^{n+1}$, i.e. a complex line through the origin in $\mathbb{C}^{n+1}$. Let

$$\gamma = \big\{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \colon\ v \in \ell \subset \mathbb{C}^{n+1}\big\}.$$

Thus, γ is a subset of the complex manifold $\mathbb{P}^n \times \mathbb{C}^{n+1}$. Show that the projection to the first component

$$\pi_1 \colon \gamma \longrightarrow \mathbb{P}^n, \qquad (\ell, v) \longrightarrow \ell,$$

defines a holomorphic line bundle. In particular, describe trivializations of γ over the open subsets

$$\mathcal{U}_i \equiv \big\{[X_0, \ldots, X_n] \colon\ X_i \neq 0\big\}$$

and compute all the transition maps $g_{ij} \colon \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow \mathbb{C} - \{0\}$; these maps should be analytic. The line bundle $\gamma \longrightarrow \mathbb{P}^n$ is called *the tautological line bundle over* $\mathbb{P}^n$.

Problem K (roughly Chapter 7, #8)

Recall that an element P of $G(2, n)$ is a complex two-dimensional linear subspace of $\mathbb{C}^n$, i.e. a complex plane through the origin in $\mathbb{C}^n$. Let

$$\gamma = \big\{(P, v) \in G(2, n) \times \mathbb{C}^n \colon\ v \in P \subset \mathbb{C}^n\big\}.$$

Thus, γ is a subset of the complex manifold $G(2, n) \times \mathbb{C}^n$. Show that the projection to the first component

$$\pi_1 \colon \gamma \longrightarrow G(2, n), \qquad (P, v) \longrightarrow P,$$

defines a vector bundle of rank 2. The vector bundle $\gamma \longrightarrow G(2, n)$ is called *the tautological two-plane bundle over* $G(2, n)$.

Discussion Problems for 11/20,27

Lines in projective spaces

Day 1: Review the definition of the Schubert cycles σ_{ab} in $G(2,n)$. Recall the formulas from the previous discussion for their intersections. What do they have to do with the multiplication rules for Young tableaux in Aaron Bertram's talk? Use these formulas to obtain all possible line counts for $\mathbb{P}^2$ and $\mathbb{P}^3$ (1+3 of them). Do Exercise 5 in Chapter 7.

Day 2: Review the definition of the Schubert cycles σ_{ab} in $G(2,n)$ and the intersection formulas for them. Describe a solution to Exercise 10 in Chapter 7. Find the number of lines that lie on a general cubic hypersurface in $\mathbb{P}^3$ and the number of lines that lie on a general quintic hypersurface in $\mathbb{P}^4$ (this is mostly done in the book, but not completely).

We should finish all of the relevant material for this homework assignment by Tuesday, 11/13. Please try to complete the written assignment and study the discussion part before Thursday, 11/15, and come to the office hours then with any questions.