MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2018

Homework Assignment III

Written Assignment due on Tuesday, 9/25, at 1pm in ESS 181
(or by 9/25, noon, in Math 3-111)

Please do 6 of the following problems: Chapter 4, #1,2,3,5, and B-D below, with B-D counted as 2 problems each; B-D are also for presentation on 9/25.

Problem B (9/25, ∼15mins)

Let $U \subset \mathbb{C}^4 \times \mathbb{C}^4$ be the subspace consisting of pairs of linearly independent vectors and $A \subset U$ of pairs of vectors that are orthonormal with respect to the standard Hermitian inner-product on $\mathbb{C}^4$. Thus, each element of U and A determines a two-dimensional linear subspace of $\mathbb{C}^4$; this induces surjective maps

$$\pi \colon U \longrightarrow G(2,4), \qquad \pi' \colon A \longrightarrow G(2,4).$$

Show that these maps induce the same topology on $G(2,4)$.

Problem C (9/25, ∼20mins)

(a) For each $i = 0, 1, \ldots, n$, let

$$U_i = \big\{ [X_0, X_1, \ldots, X_n] \in \mathbb{C}P^n \colon\ X_i \neq 0 \big\},$$
$$\phi_i \colon U_i \longrightarrow \mathbb{C}^n, \quad [X_0, X_1, \ldots, X_n] \longrightarrow (X_0/X_i, X_1/X_i, \ldots, X_{i-1}/X_i, X_{i+1}/X_i, \ldots, X_n/X_i).$$

Show that the maps ϕ_i are homeomorphisms, while

$$\phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

are analytic/holomorphic maps between open subspaces of $\mathbb{C}^n$.

(b) Describe manifold charts for $G(2,4)$ (for the quotient topology of Problem B) and show that $G(2,4)$ is also a complex manifold (the overlap maps are analytic/holomorphic).

Problem D (9/25, ∼15mins)

Let $F = F(X_0, \ldots, X_n)$ be a homogeneous polynomial of degree $d \in \mathbb{Z}^+$ and $(Y_0, \ldots, Y_n) \in \mathbb{C}^{n+1} - 0$. If $F(Y_0, \ldots, Y_n) = 0$, but $\frac{\partial F}{\partial X_i}\big|_{(Y_0,\ldots,Y_n)} \neq 0$ for some $i = 0, 1, \ldots, n$, show that the hypersurface $Z(F) \subset \mathbb{C}P^n$ is a complex manifold in a neighborhood of $[Y_0, \ldots, Y_n] \in Z(F)$. Show that the tangent hyperplane to $Z(F)$ at $[Y_0, \ldots, Y_n]$ is given by the equation

$$G(X_0, \ldots, X_n) = \frac{\partial F}{\partial X_0}\Big|_{(Y_0,\ldots,Y_n)} X_0 + \ldots + \frac{\partial F}{\partial X_n}\Big|_{(Y_0,\ldots,Y_n)} X_n = 0.$$

Further Discussion Problems for 9/25, 10/2

Bezoit's Theorem for $\mathbb{C}P^2$*:* If $C,D\subset\mathbb{C}P^2$ are curves of degrees $c,d\in\mathbb{Z}^+$ such that $C\cap D$ is finite, then the cardinality of the set $C\cap D$ counted with multiplicity $m_p(C,D)\in\mathbb{Z}^+$ for each point $p\in C\cap D$ is cd.

The number $m_p(C,D)$ is defined so that if the curves C and D are deformed slightly and generically (by deforming the homogeneous polynomials defining C and D), then $m_p(C,D)$ is the number of points in the intersection of the deformed curves that lie near p. Thus, the weighted cardinality of $C\cap D$ does not change under small changes in (C,D). It thus must be independent of C and D provided the space

$$\mathfrak{X}_{c,d}\equiv\big\{(F,G)\in HP_c(\mathbb{C}^3)\times HP_d(\mathbb{C}^3)\colon Z(F)\cap Z(G)\text{ is finite}\big\}$$

is connected (as in class $HP_c(\mathbb{C}^3)$ is the space of homogeneous polynomials on $\mathbb{C}^3$ of degree c). We can thus determine the weighted cardinality of $C\cap D$ by determining it for a specific pair in $\mathfrak{X}_{c,d}$; in class, C and D were taken to consist of c and d lines, respectively, with all $c+d$ lines being distinct, obtaining Bezoit's Theorem. The aim of this discussion problem is to fill in some of the gaps in the argument.

Part I (9/25, ~20mins): A topological space $\mathfrak{X}$ is called connected if it can't be written as a disjoint union of two nonempty open subset, $\mathfrak{X}\neq U\sqcup V$; $\mathfrak{X}$ is called path-connected if for any $p,q\in\mathfrak{X}$ there exists a continuous map $f\colon[0,1]\longrightarrow\mathfrak{X}$ such that $f(0)=p$ and $f(1)=q$ (thus every two points in $\mathfrak{X}$ are connected by a path). Show that
(a) any continuous map from a connected space to $\mathbb{Z}$ is constant;
(b) any path-connected space is connected and thus $\mathbb{C}^n$ is connected;
(c) if $A\subset\mathfrak{X}$ is connected (in the subspace topology) and $A\subset B\subset\overline{A}$, then $B\subset\mathfrak{X}$ is also connected.

Part II (10/2, ~30mins): Show that
(a) if $f\colon\mathbb{C}^n\longrightarrow\mathbb{C}$ is a non-constant analytic function (analytic in each variable), then for every $p\in f^{-1}(0)$ there exists $r>0$ such that $B_r(p)-f^{-1}(0)$, where $B_r(p)$ is the r-ball centered at p, is path-connected;
(b) if $f\colon\mathbb{C}^n\longrightarrow\mathbb{C}$ is a non-constant analytic function, then $\mathbb{C}^n-f^{-1}(0)$ is path-connected;
(c) $\mathfrak{X}_{c,d}$ is connected.

Part III (10/2, ~40mins): Let $f,g:\mathbb{C}^2\longrightarrow\mathbb{C}$ be two polynomials of degrees at most c and d (not necessarily homogeneous) such that $f(0),g(0)=0$ and there exists $r>0$ such that $B_r(0)-0$ contains no points $f^{-1}(0)\cap g^{-1}(0)$. For $p\in\mathbb{C}^2$, let $\nabla f|_p\colon\mathbb{C}^2\longrightarrow\mathbb{C}$ be the gradient of f at p. If $c\in\mathbb{Z}^+$, denote by $P_c(\mathbb{C}^2)$ the space of polynomials on $\mathbb{C}^2$ of degree at most c; there is a natural norm on $P_c(\mathbb{C}^2)$, since such polynomials correspond to tuples of elements of $\mathbb{C}$. Show that
(a) if $(\ker\nabla f|_0)\cap(\ker\nabla g|_0)=\{0\}$, there exists $\epsilon>0$ such that the set

$$\{f+\tilde{f}\}^{-1}(0)\cap\{g+\tilde{g}\}^{-1}(0)\cap B_{r/2}(0)\subset\mathbb{C}^2$$

consists of precisely one element whenever $|\tilde{f}|,|\tilde{g}|<\epsilon$ (*in such a case,* 0 *is said to be a simple intersection point, or point of intersection multiplicity* 1*, of* $f^{-1}(0)$ *and* $g^{-1}(0)$);

(b) the set
$$\mathfrak{X}_{c,d}(f,g)\equiv\Big\{(\tilde{f},\tilde{g})\in P_c(\mathbb{C}^2)\times P_d(\mathbb{C}^2)\colon\ Z(f+\tilde{f})\cap Z(g+\tilde{g})\text{ is finite};\ (\ker\nabla(f+\tilde{f})|_p)\cap(\ker\nabla(g+\tilde{g})|_p)=\{0\}\ \forall\,p\in Z(f+\tilde{f})\cap Z(g+\tilde{g})\Big\}$$

is connected and non-empty;
(c) there exists $\epsilon>0$ such that the cardinality of the set

$$\{f+\tilde{f}\}^{-1}(0)\cap\{g+\tilde{g}\}^{-1}(0)\cap B_{r/2}(0)\subset\mathbb{C}^2$$

is independent of $\tilde{f},\tilde{g}\in\mathfrak{X}_{c,d}(f,g)$ with $|\tilde{f}|,|\tilde{g}|<\epsilon$. (*This number is called the multiplicity of* $0\in\mathbb{C}^2$ *as an intersection point of the curves* $Z(f)$ *and* $Z(g)$ *and is denoted by* $m_0(f,g)$ *or* $m_0(Z(f),Z(g))$).