MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 9

Problem 1 (8pts)

Let $(X,\mathcal{F},\mu)$ be a measure space. Suppose there exist $A,B\in\mathcal{F}$ such that $A\cap B=\emptyset$ and $\mu(A),\mu(B)\in\mathbb{R}^+$. Show that the norm $\|\cdot\|_p$ on $L^p(X)$ is not induced by an inner-product on $L^p(X)$ for any $p\in[1,\infty]-\{2\}$.

Let $f=\mathbb{1}_A$ and $g=\mathbb{1}_B$. Thus,

$$\|f\|_p=\mu(A)^{1/p},\quad \|g\|_p=\mu(B)^{1/p},\quad \|f+g\|_p,\|f-g\|_p=\big(\mu(A)+\mu(B)\big)^{1/p}.$$

If $\|\cdot\|_p$ is induced by an inner-product on $L^p(X)$, then

$$\|f+g\|_p^2+\|f-g\|_p^2=2\big(\|f\|_p^2+\|g\|_p^2\big),\qquad \big(\mu(A)+\mu(B)\big)^{2/p}=\mu(A)^{2/p}+\mu(B)^{2/p}.$$

If $2/p<1$ (resp. $2/p>1$), then the left-hand above is smaller (resp. larger) than the right-hand side. Thus, $\|\cdot\|_p$ is not induced by an inner-product on $L^p(X)$ if $p\neq 2$.

Problem 2 (15pts)

Let $C^{0,2}(\mathbb{R})\subset L^2(\mathbb{R})$ denote the subspace of continuous square-integrable functions. Define

$$L^2_0(\mathbb{R})=\big\{f\in L^2(\mathbb{R})\colon f=0\text{ a.e. on }[0,1]\big\},\qquad C^{0,2}_0(\mathbb{R})=L^2_0(\mathbb{R})\cap C^{0,2}(\mathbb{R}).$$

(a) *Let $f\in C^{0,2}(\mathbb{R})$ be such that $f(x)\neq 0$ for some $x\in\mathbb{R}-[0,1]$. Show that there exists $g\in C^{0,2}_0(\mathbb{R})$ such that $\langle\!\langle f,g\rangle\!\rangle_2\neq 0$. Conclude that $f\in C^{0,2}(\mathbb{R})$ has a projection to $C^{0,2}_0(\mathbb{R})$ if and only if $f(0)=f(1)=0$.*

(b) *Let $f\in L^2(\mathbb{R})$. Determine the projection of f to $L^2_0(\mathbb{R})$.*

(a; **10pts**) Let $f\in C^{0,2}(\mathbb{R})$ and $x^*\in\mathbb{R}-[0,1]$ be such that $f(x^*)\neq 0$. By multiplying f by -1 if necessary, we can assume that $f(x^*)>0$. Since f is continuous, there exists $\delta>0$ such that $f(x)>0$ for all $x\in(x^*-\delta,x^*+\delta)$. Since $x^*\notin[0,1]$, by shrinking δ we can assume that

$$\big(x^*-\delta,x^*+\delta\big)\cap[0,1]=\emptyset.$$

Define $g\in C^{0,2}_0(\mathbb{R})$ by

$$g\colon\mathbb{R}\longrightarrow\mathbb{R}^{\ge 0},\qquad g(x)=\begin{cases}0, & \text{if } x\in\mathbb{R}-(x^*-\delta,x^*+\delta);\\ \delta^2-(x-x^*)^2, & \text{if } x\in[x^*-\delta,x^*+\delta].\end{cases}$$

Thus,

$$\langle\!\langle f,g\rangle\!\rangle_2=\int_{x^*-\delta}^{x^*+\delta}fg\,\mathrm{d}x>0,$$

because fg is a continuous positive function on $(x^*-\delta,x^*+\delta)$.

Suppose $f\in C^{0,2}(\mathbb{R})$ and $f_0\in C^{0,2}_0(\mathbb{R})$ is its projection. Thus, $\langle\!\langle f-f_0,g\rangle\!\rangle_2=0$ for every $g\in C^{0,2}_0(\mathbb{R})$. The previous paragraph then implies that $f(x)=f_0(x)$ for every $x\in\mathbb{R}-[0,1]$. By the continuity of f, this implies that $f(0)=f_0(0)$ and $f(1)=f_0(1)$. Since $f_0\in C^{0,2}_0(\mathbb{R})$ is continuous and vanishes almost everywhere on $[0,1]$, it in fact vanishes everywhere on $[0,1]$. In particular, $f_0(0)=f_0(1)=0$. Combining this with the previous conclusion, we obtain $f(0)=f(1)=0$.

Suppose $f\in C^{0,2}(\mathbb{R})$ and $f(0)=f(1)=0$. Define

$$f_0\colon \mathbb{R}\longrightarrow\mathbb{R}, \qquad f_0(x)=\begin{cases}0, & \text{if } x\in[0,1];\\ f(x), & \text{if } x\in\mathbb{R}-(0,1).\end{cases}$$

By the assumption $f(0)=f(1)=0$, this function is well-defined. It is continuous because it is continuous on two closed sets whose union is its domain. Thus, $f_0\in C_0^{0,2}(\mathbb{R})$. Since $f(x)=f_0(x)$ for every $x\in\mathbb{R}-[0,1]$, $\langle\!\langle f-f_0,g\rangle\!\rangle_2=0$ for every $g\in C_0^{0,2}(\mathbb{R})$. Thus, f_0 is the projection of f to $C_0^{0,2}(\mathbb{R})$.

(b; **5pts**) Let $f:\mathbb{R}\longrightarrow\overline{\mathbb{R}}$ be a representative of an element of $L^2(\mathbb{R})$ (an element of $L^2(\mathbb{R})$ is an equivalence class). Let

$$f_0=\mathbb{1}_{\mathbb{R}-[0,1]}f\colon\mathbb{R}\longrightarrow\mathbb{R}.$$

Thus, $f_0\in L_0^2(\mathbb{R})$ and $f(x)=f_0(x)$ for every $x\in\mathbb{R}-[0,1]$. The latter implies that $\langle\!\langle f-f_0,g\rangle\!\rangle_2=0$ for every $g\in L_0^2(\mathbb{R})$. Therefore, the element in $L^2(\mathbb{R})$ represented by f_0 is the projection of the element represented by f to $L_0^2(\mathbb{R})$.

Problem 3 (17pts)

(a) *Let X_1,X_2 be sets, $\sigma(\mathcal{S}_1)$ be the σ-field on X_1 generated by a collection $\mathcal{S}_1\subset 2^{X_1}$ of subsets of X_1, and $\sigma(\mathcal{S}_2)$ be the σ-field on X_2 generated by a collection $\mathcal{S}_2\subset 2^{X_2}$ of subsets of X_2. Show that the σ-fields $\sigma(S_1\times S_2)$ and $\sigma(\sigma(\mathcal{S}_1)\times\sigma(\mathcal{S}_2))$ on $X_1\times X_2$ generated by the collections*

$$\mathcal{S}_1\times\mathcal{S}_2=\big\{A\times B\colon A\in\mathcal{S}_1,\, B\in\mathcal{S}_2\big\} \qquad \textit{and}$$
$$\sigma(\mathcal{S}_1)\times\sigma(\mathcal{S}_2)=\big\{A\times B\colon A\in\sigma(\mathcal{S}_1),\, B\in\sigma(\mathcal{S}_2)\big\},$$

respectively, are the same.

(b) *For $n\in\mathbb{Z}^+$, let $\mathcal{M}_n\subset 2^{\mathbb{R}^n}$ be the collection of Lebesgue measurable subsets as described in Section 6.1. Show that*

$$\sigma\big(\mathcal{M}_{n_1}\times\mathcal{M}_{n_2}\big)\subsetneq\mathcal{M}_{n_1+n_2} \qquad \forall\, n_1,n_2\in\mathbb{Z}^+;$$

note the inequality above.

(a; **9pts**) Since $\mathcal{S}_1\times\mathcal{S}_2\subset\sigma(\mathcal{S}_1)\times\sigma(\mathcal{S}_2)$,

$$\sigma\big(S_1\times S_2\big)\subset\sigma\big(\sigma(\mathcal{S}_1)\times\sigma(\mathcal{S}_2)\big).$$

We need to show the opposite inclusion. Let

$$\mathcal{F}_1=\big\{A\in\sigma(S_1)\colon A\times X_2\in\sigma\big(S_1\times S_2\big)\big\}, \qquad \mathcal{F}_2=\big\{B\in\sigma(S_2)\colon X_1\times B\in\sigma\big(S_1\times S_2\big)\big\}.$$

Since $X_1\times X_2\in\sigma(S_1\times S_2)$, $X_1\in\mathcal{F}_1$. Since

$$X_1\times X_2-A\times X_2=(X_1-A)\times X_2$$

and the collection $\sigma(S_1\times S_2)$ is closed under complements, the collection $\mathcal{F}_1$ is also closed under complements (if $A\in\mathcal{F}_1$, then $A^c\in\mathcal{F}_1$). Since

$$\bigcup_{n=1}^{\infty}(A_n\times X_2)=\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)\times X_2$$

and the collection $\sigma(S_1 \times S_2)$ is closed under countable unions, the collection $\mathcal{F}_1$ is also closed under countable unions (if $A_1, A_2, \ldots \in \mathcal{F}_1$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_1$). Thus, $\mathcal{F}_1$ is a σ-field on X_1. By definition of $\mathcal{F}_1$,

$$S_1 \subset \mathcal{F}_1 \subset \sigma(S_1).$$

Since $\sigma(\mathcal{F}_1)$ is the smallest σ-field containing S_1, it follows that $\mathcal{F}_1 = \sigma(S_1)$. By the same reasoning, $\mathcal{F}_2 = \sigma(S_2)$. Thus,

$$\big\{A \times X_2 \colon A \in \sigma(\mathcal{F}_1)\big\}, \big\{X_1 \times B \colon B \in \sigma(\mathcal{F}_2)\big\} \subset \sigma(S_1 \times S_2).$$

Since the collection $\sigma(S_1 \times S_2)$ is closed under pairwise (and more generally countable) intersections, it follows that

$$\sigma(S_1) \times \sigma(S_2) \equiv \big\{(A \times X_2) \cap (X_1 \times B) \colon A \in \sigma(\mathcal{F}_1), B \in \sigma(\mathcal{F}_2)\big\} \subset \sigma\big(S_1 \times S_2\big) \subset \sigma\big(\sigma(S_1) \times \sigma(S_2)\big).$$

Since $\sigma(\sigma(S_1) \times \sigma(S_2))$ is the smallest σ-field on $X_1 \times X_2$ containing $\sigma(S_1) \times \sigma(S_2)$, it follows that the last inclusion above is in fact an equality.

(b; **8pts**) The collection $\mathcal{M}_n \subset 2^{\mathbb{R}^n}$ consists of the subsets $E \subset \mathbb{R}^n$ that satisfy (2.6) in the book with the outer measure $m^* \equiv m_n^*$ as in Definition 2.3 with the intervals and their lengths replaced by n-dimensional "rectangles" and their volumes. This collection contains all n-dimensional "rectangles" and all m_n^*-null subsets of $\mathbb{R}^n$. The former implies that $\mathcal{M}_n$ contains the σ-field $\mathcal{B}_n$ generated by the collection of n-dimensional "rectangles". If $E \subset [0,1]$ is a non-measurable subset as on p302, then

$$E_n \equiv E \times [0,1]^{n-1} \subset \mathbb{R}^n$$

is a non-measurable subset with respect to m_n^* by the same reasoning as on p302.

Let $n_1, n_2 \in \mathbb{Z}^+$. It is fairly immediate from the definition that

$$m_{n_1+n_2}^*(A_1 \times A_2) \le m_{n_1}^*(A_1) \cdot m_{n_2}^*(A_2) \qquad \forall A_1 \subset \mathbb{R}^{n_1},\ A_2 \subset \mathbb{R}^{n_2}.$$

If $E_1 \in \mathcal{M}_{n_1}$ and $E_2 \in \mathcal{M}_{n_2}$, there exist

$$B_1, F_1 \in \mathcal{M}_{n_1} \quad \text{and} \quad B_2, F_2 \in \mathcal{M}_{n_2} \qquad \text{s.t.}$$
$$E_1 = B_1 \cup F_1, \quad B_1 \in \mathcal{B}_{n_1}, \quad m_{n_1}^*(F_1) = 0, \qquad E_2 = B_2 \cup F_2, \quad B_2 \in \mathcal{B}_{n_2}, \quad m_{n_2}^*(F_2) = 0.$$

By (a), $B_1 \times B_2 \in \mathcal{B}_{n_1+n_2}$ and thus $B_1 \times B_2 \in \mathcal{M}_{n_1+n_2}$. By the above inequality, $B_1 \times F_2$, $F_1 \times B_2$, and $F_1 \times F_2$ are $m_{n_1+n_2}^*$-null subsets of $\mathbb{R}^{n_1+n_2}$ and thus belong to $\mathcal{M}_{n_1+n_2}$. Since $\mathcal{M}_{n_1+n_2}$ is closed under finite (and more generally countable) unions, it follows that

$$E_1 \times E_2 = B_1 \times B_2 \cup B_1 \times F_2 \cup B_1 \times F_2 \cup F_1 \times F_2 \in \mathcal{M}_{n_1+n_2}.$$

Since $\sigma(\mathcal{M}_{n_1} \times \mathcal{M}_{n_2})$ is the smallest σ-field containing $\mathcal{M}_{n_1} \times \mathcal{M}_{n_2}$, we conclude that

$$\sigma\big(\mathcal{M}_{n_1} \times \mathcal{M}_{n_2}\big) \subset \mathcal{M}_{n_1+n_2};$$

If $E \subset \mathbb{R}^{n_1}$, then $E \times \{0^{n_2}\}$ is $m_{n_1+n_2}^*$-null and thus belongs to $\mathcal{M}_{n_1+n_2}$. Since

$$\big(E \times \{0^{n_2}\}\big)_{0^{n_2}} = E,$$

Theorem 6.4 implies that $E \times \{0^{n_2}\}$ does not belong to $\sigma(\mathcal{M}_{n_1} \times \mathcal{M}_{n_2})$ if E is not $m_{n_1}^*$-measurable. Thus,

$$\sigma\big(\mathcal{M}_{n_1} \times \mathcal{M}_{n_2}\big) \not\supset \mathcal{M}_{n_1+n_2}.$$