MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 7

Problem 1 (8pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $p,q,r\in[1,\infty]$ be such that $1/p+1/q=1/r$. Show that

$$\big\|fg\big\|_r \le \|f\|_p\|g\|_q$$

for all measurable functions $f,g\colon X\longrightarrow\mathbb{R}$ (cases with $p,q,r=\infty$ may require separate treatment).

If $p=\infty$, then $q=r$ and

$$\big\|fg\big\|_r \equiv \big\||f|\cdot|g|\big\|_q \le \big\|\|f\|_\infty|g|\big\|_q = \|f\|_p\|g\|_q\,.$$

If $r=\infty$, then $p,q=\infty$. Thus, it remains to consider the case $p,q,r<\infty$. Let $p'=p/r$ and $q'=q/r$. By the assumption on p,q,r, $1/p'+1/q'=1$. By Hölder's Inequality,

$$\big\|fg\big\|_r \equiv \big(\big\||f|^r|g|^r\big\|_1\big)^{1/r} \le \big(\big\||f|^r\big\|_{p'}\big\||g|^r\big\|_{q'}\big)^{1/r} \equiv \big(\big\||f|^{rp'}\big\|_1^{r/p}\big\||g|^{rq'}\big\|_1^{r/q}\big)^{1/r} = \|f\|_p\|g\|_q\,.$$

Problem 2 (12pts)

Find a function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ which is in $L^2(\mathbb{R}^+)$, but not in $L^p(\mathbb{R}^+)$ for any $p\in[1,\infty]-\{2\}$. Justify your answer.

For each $n\in\mathbb{Z}^+$, define

$$f_n\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+, \qquad f_n(x)=\begin{cases} x^{-\frac{1}{2}+\frac{1}{2n}}, & \text{if } x\le 1;\\ x^{-\frac{1}{2}-\frac{1}{2n}}, & \text{if } x\ge 1.\end{cases}$$

By the Ratio Test, the sum

$$f(x)\equiv\sum_{n=1}^{\infty}2^{-n}f_n(x)$$

converges for every $x\in\mathbb{R}^+$. By the Monotone Convergence Theorem and Minkowski's Inequality,

$$\begin{aligned}\|f\|_2^2 &\equiv \int_{\mathbb{R}^+}\bigg(\lim_{k\longrightarrow\infty}\sum_{n=1}^{n=k}2^{-n}f_n\bigg)^2\mathrm{d}m = \lim_{k\longrightarrow\infty}\int_{\mathbb{R}^+}\bigg(\sum_{n=1}^{n=k}2^{-n}f_n\bigg)^2\mathrm{d}m\\ &= \lim_{k\longrightarrow\infty}\bigg\|\sum_{n=1}^{n=k}2^{-n}f_n\bigg\|_2^2 \le \bigg(\lim_{k\longrightarrow\infty}\sum_{n=1}^{n=k}\Big\|2^{-n}f_n\Big\|_2\bigg)^2 = \bigg(\sum_{n=1}^{\infty}2^{-n}\|f_n\|_2\bigg)^2.\end{aligned}$$

Since $\|f_n\|_2=\sqrt{2n}$, it follows that

$$\|f\|_2 \le \sum_{n=1}^{\infty}2^{-n}\sqrt{2n}<\infty\,.$$

Thus, $f\in L^2(\mathbb{R}^+)$.

Since $f_1(x)\longrightarrow\infty$ for as $x\longrightarrow 0$ and $f\geq f_1$, $\|f\|_\infty=\infty$ and $f\notin L^\infty(\mathbb{R}^+)$. If $p\in[1,2)$, there exists $n\in\mathbb{Z}^+$ so that $(1/2+1/2n)p<1$ and thus

$$\|f\|_p^p \geq \|f_n\|_p^p \geq \int_{(1,\infty)} x^{-(\frac{1}{2}+\frac{1}{2n})p}\mathrm{d}m \geq \int_{(1,\infty)} x^{-1}\mathrm{d}m = \int_1^\infty x^{-1}\mathrm{d}x = \infty.$$

Thus, $f\notin L^p(\mathbb{R}^+)$. If $p\in(2,\infty)$, there exists $n\in\mathbb{Z}^+$ so that $(1/2-1/2n)p>1$ and thus

$$\|f\|_p^p \geq \|f_n\|_p^p \geq \int_{(0,1)} x^{-(\frac{1}{2}-\frac{1}{2n})p}\mathrm{d}m \geq \int_{(0,1)} x^{-1}\mathrm{d}m = \int_0^1 x^{-1}\mathrm{d}x = \infty.$$

Thus, $f\notin L^p(\mathbb{R}^+)$. The equalities of Lebesgue and Riemann integrals in the two computations above hold because $x^{-1}>0$. This is also used to compute $\|f_n\|_2$ above.

Problem 3 (10pts)

For each of the following sequences of measurable functions $f_1,f_2,\ldots:\mathbb{R}^+\longrightarrow\mathbb{R}$, determine whether it (the sequence) lies in $L^1(\mathbb{R}^+)$, $L^2(\mathbb{R}^+)$, and if so whether it is Cauchy in there (this is between 2 and 4 questions for each sequence below). Justify your answers.

(a) $f_n=\mathbb{1}_{(0,n)}/\sqrt{x}$ (b) $f_n=\mathbb{1}_{(0,n)}/(x+1)$

(a; **5pts**) Since

$$\begin{aligned}\|f_n\|_1 &\equiv \int_{\mathbb{R}^+}|f_n|\mathrm{d}m = \int_{(0,n)} 1/\sqrt{x}\,\mathrm{d}m = \int_0^n 1/\sqrt{x}\,\mathrm{d}x = 2\sqrt{n} < \infty,\\ \|f_n\|_2^2 &\equiv \int_{\mathbb{R}^+}|f_n|^2\mathrm{d}m = \int_{(0,n)} 1/x\,\mathrm{d}m = \int_0^n 1/x\,\mathrm{d}x = \infty,\end{aligned}$$

$f_n\in L^1(\mathbb{R}^+)$ and $f_n\notin L^2(\mathbb{R}^+)$. Let $k,n\in\mathbb{Z}^+$ with $k\leq n$. Since

$$\begin{aligned}\big\|f_k-f_n\big\|_1 &\equiv \int_{\mathbb{R}^+}|f_k-f_n|\mathrm{d}m = \int_{[k,n]} 1/\sqrt{x}\,\mathrm{d}m = \int_k^n 1/\sqrt{x}\,\mathrm{d}m\\ &= 2\big(\sqrt{n}-\sqrt{k}\big)\longrightarrow\infty \qquad \text{as } n\longrightarrow\infty,\end{aligned}$$

the sequence $f_1,f_2,\ldots$ is not Cauchy in $L^1(\mathbb{R}^+)$. Alternatively, $f_n\longrightarrow f\equiv 1/\sqrt{x}$ pointwise. Since $f\notin L^1(\mathbb{R}^+)$, the sequence $f_1,f_2,\ldots$ in not Cauchy in $L^1(\mathbb{R}^+)$.

(b; **5pts**) Since

$$\begin{aligned}\|f_n\|_1 &\equiv \int_{\mathbb{R}^+}|f_n|\mathrm{d}m = \int_{(0,n)} 1/(x+1)\,\mathrm{d}m = \int_0^n 1/(x+1)\,\mathrm{d}x = \ln(x) < \infty,\\ \|f_n\|_2^2 &\equiv \int_{\mathbb{R}^+}|f_n|^2\mathrm{d}m = \int_{(0,n)} 1/(x+1)^2\,\mathrm{d}m = \int_0^n 1/(x+1)^2\,\mathrm{d}x = 1-\frac{1}{n+1} < \infty,\end{aligned}$$

$f_n \in L^1(\mathbb{R}^+), L^2(\mathbb{R}^+)$. Let $k, n \in \mathbb{Z}^+$ with $k \le n$. Since

$$\begin{aligned}
\|f_k - f_n\|_1 &\equiv \int_{\mathbb{R}^+} |f_k - f_n| \mathrm{d}m = \int_{[k,n]} 1/(x+1)\,\mathrm{d}m = \int_k^n 1/(x+1)\,\mathrm{d}m \\
&= \ln(n+1) - \ln(k+1) \longrightarrow \infty \qquad \text{as } \ n \longrightarrow \infty, \\
\|f_k - f_n\|_2^2 &\equiv \int_{\mathbb{R}^+} |f_k - f_n|^2 \mathrm{d}m = \int_{[k,n]} 1/(x+1)^2\,\mathrm{d}m = \int_k^n 1/(x+1)^2\,\mathrm{d}m \\
&= 1/(k+1) - 1/(n+1) \le 1/(k+1) \quad \forall\, n \ge k,
\end{aligned}$$

the sequence $f_1, f_2, \ldots$ is not Cauchy in $L^1(\mathbb{R}^+)$ and is Cauchy in $L^2(\mathbb{R}^+)$. Alternatively,

$$f_n \longrightarrow f \equiv 1/(1+x)$$

pointwise. Since $f \notin L^1(\mathbb{R}^+)$, the sequence $f_1, f_2, \ldots$ is not Cauchy in $L^1(\mathbb{R}^+)$. While $f \in L^2(\mathbb{R}^+)$, this cannot be used to conclude that the sequence $f_1, f_2, \ldots$ is Cauchy in $L^2(\mathbb{R}^+)$.

Problem 4 (20pts)

Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. For a differentiable function $f: \mathbb{R} \longrightarrow \mathbb{R}$, define

$$\|f\|_{p,1} = \|f\|_p + \|f'\|_p \,.$$

(a) *Show that this defines a norm on the vector space*

$$C^{1,p}(\mathbb{R}) \equiv \big\{ f \in C^1(\mathbb{R}) \colon \|f\|_{p,1} < \infty \big\}.$$

Do not forget to justify why $C^{1,p}(\mathbb{R})$ is a vector space ($C^1(\mathbb{R})$ is the space of differentiable functions $f: \mathbb{R} \longrightarrow \mathbb{R}$).

(b) *Show that*

$$\big|f(x) - f(y)\big| \le \|f'\|_p |x-y|^{1/q}, \qquad \left| f(x) - \frac{1}{2}\int_{x-1}^{x+1} f(y)\mathrm{d}y \right| \le \|f'\|_p, \qquad \|f\|_\infty \le \|f\|_{p,1}$$

for all $f \in C^{1,p}(\mathbb{R})$ and $x, y \in \mathbb{R}$ (cases with $p, q = \infty$ may require separate treatment).

(c) *Let $f_1, f_2, \ldots \in C^{1,p}(\mathbb{R})$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{p,1}$. Show that it converges uniformly to a bounded continuous function $f: \mathbb{R} \longrightarrow \mathbb{R}$.*

(a; **5pts**) If $f \in C^{1,p}(\mathbb{R})$ and $c \in \mathbb{R}$, then

$$\|cf\|_{p,1} \equiv \|cf\|_p + \|cf'\|_p = |c|\|f\|_p + |c|\|f'\|_p \equiv |c|\|f\|_{p,1} < \infty; \tag{1}$$

the equality in the middle holds because $\|\cdot\|_p$ satisfies this on $\mathcal{L}^p(X)$. Thus, $cf \in C^{1,p}(\mathbb{R})$. If $f, g \in C^{1,p}(\mathbb{R})$,

$$\|f+g\|_{p,1} \equiv \|f+g\|_p + \|f'+g'\|_p \le \big(\|f\|_p + \|g\|_p\big) + \big(\|f'\|_p + \|g'\|_p\big) \equiv \|f\|_{p,1} + \|g\|_{p,1}\,; \tag{2}$$

the inequality in the middle holds by Minkowski's inequality. Thus, $f+g\in C^{1,p}(\mathbb{R})$. We conclude that $C^{1,p}(\mathbb{R})$ is a vector space. By (1) and (2), the function

$$\|\cdot\|_p\colon C^{1,p}(\mathbb{R}) \longrightarrow \mathbb{R}^{\ge 0} \tag{3}$$

satisfies two of the three properties required of a norm. If $f\in C^{1,p}(\mathbb{R})$ and $\|f\|_{p,1}=0$, then $\|f\|_p=0$ and so $f=0$ a.e. on $\mathbb{R}$. Since f is continuous (because it is differentiable), it follows that $f=0$ and so the map (3) also satisfies the remaining property required of a norm.

(b; **10pts**) Let $f\in C^{1,p}(\mathbb{R})$ and $x,y\in\mathbb{R}$ with $x<y$. By the Fundamental Theorem of Calculus and Hölder's Inequality,

$$\begin{aligned}\big|f(y)-f(x)\big| &\le \left|\int_x^y f'(t)\mathrm{d}t\right| \le \int_x^y \big|f'(t)\big|\mathrm{d}t = \int_{[x,y]}|f'|\mathrm{d}m \equiv \big\|f'\cdot\mathbb{1}_{[x,y]}\big\|_1\\ &\le \|f'\|_p\big\|\mathbb{1}_{[x,y]}\big\|_q = \|f'\|_p|x-y|^{1/q}\,.\end{aligned}$$

This establishes the first inequality. Using it, we obtain

$$\begin{aligned}\left|f(x)-\frac{1}{2}\int_{x-1}^{x+1}f(y)\mathrm{d}y\right| &= \frac{1}{2}\left|\int_{x-1}^{x+1}\big(f(x)-f(y)\big)\mathrm{d}y\right| \le \frac{1}{2}\int_{x-1}^{x+1}\big|f(x)-f(y)\big|\mathrm{d}y\\ &\le \frac{1}{2}\int_{x-1}^{x+1}\|f'\|_p|x-y|^{1/q}\mathrm{d}y = \|f'\|_p\int_0^1 r^{1/q}\mathrm{d}r = \|f'\|_p\cdot\frac{q}{q+1} \le \|f'\|_p\,.\end{aligned}$$

This establishes the second inequality. Using it and Hölder's Inequality, we obtain

$$\begin{aligned}\big|f(x)\big| &\le \left|f(x)-\frac{1}{2}\int_{x-1}^{x+1}f(y)\mathrm{d}y\right| + \left|\frac{1}{2}\int_{x-1}^{x+1}f(y)\mathrm{d}y\right| \le \|f'\|_p + \frac{1}{2}\big\|f\cdot\mathbb{1}_{[x-1,x+1]}\big\|_1\\ &\le \|f'\|_p + \frac{1}{2}\|f\|_p\big\|\mathbb{1}_{[x-1,x+1]}\big|_q = \|f'\|_p + \frac{1}{2}\|f\|_p\cdot 2^{1/q} \le \|f'\|_p+\|f\|_p \equiv \|f\|_{p,1}\,.\end{aligned}$$

This establishes the third inequality.

(c; **5pts**) By the last inequality in (b) applied to the continuous functions f_n and f_m-f_n, each function f_n is bounded and the sequence $f_1,f_2,\ldots$ is Cauchy with respect to the sup-norm on the space of continuous functions. In particular, the sequence $f_1(x),f_2(x),\ldots$ is Cauchy in $\mathbb{R}$ for every $x\in\mathbb{R}$ and thus converges to some $f(x)\in\mathbb{R}$. This defines a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ so that $f_n\longrightarrow f$ pointwise on $\mathbb{R}$. Since the sequence $f_1,f_2,\ldots$ is Cauchy with respect to the sup-norm, this convergence is uniform. Since each function f_n is continuous and bounded, it follows that so is f.