MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 6

Problem 1 (8pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $f_n\colon X\longrightarrow\overline{\mathbb{R}}$ be a sequence of measurable functions converging almost everywhere to a function f. Suppose

$$\limsup_{m\longrightarrow\infty}\bigg(\int_X\big(\sup_{n\le m}|f_n|\big)\mathrm{d}\mu\bigg)<\infty. \tag{1}$$

Show that $\displaystyle\int f\mathrm{d}\mu=\lim_{n\longrightarrow\infty}\bigg(\int f_n\mathrm{d}\mu\bigg)$.

The sequence of measurable functions

$$g_m\equiv\sup_{n\le m}|f_n|,\qquad g_m(x)=\sup\big\{|f_n(x)|\colon n=1,\ldots,m\big\},$$

is non-decreasing and thus converges to some measurable function $g\colon X\longrightarrow\mathbb{R}$. Furthermore, $g_m\ge0$ for all m. By the Monotone Convergence Theorem and (1),

$$\int_X g\,\mathrm{d}\mu\equiv\int_X\big(\lim_{m\longrightarrow\infty}g_m\big)\mathrm{d}\mu=\lim_{m\longrightarrow\infty}\bigg(\int_X g_m\mathrm{d}\mu\bigg)=\limsup_{m\longrightarrow\infty}\bigg(\int_X\big(\sup_{n\le m}|f_n|\big)\mathrm{d}\mu\bigg)<\infty.$$

Thus, $g\in\mathcal{L}^1(X)$. By the definition of g, $|f_n|\le g$ for all n. The desired statement now follows from the Dominated Convergence Theorem.

Problem 2 (17pts)

Let $(X,\mathcal{F},\mu)$ be a measure space. Suppose $f_1,f_2,\ldots\colon X\longrightarrow\overline{\mathbb{R}}$ is a sequence of measurable functions converging a.e. to a measurable function f and $g_1,g_2,\ldots\colon X\longrightarrow\overline{\mathbb{R}}$ is a sequence of integrable functions converging a.e. to an integrable function g such that

$$|f_n|\le g_n\ \text{a.e.}\qquad\text{and}\qquad\int_X g\mathrm{d}\mu=\lim_{n\longrightarrow\infty}\bigg(\int_X g_n\mathrm{d}\mu\bigg). \tag{2}$$

Show that

$$\int_X f\mathrm{d}\mu=\lim_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg). \tag{3}$$

Suppose first that $f_n\ge0$ for all n. By Fatou's Lemma,

$$\int_X f\,\mathrm{d}\mu=\int_X\big(\lim_{n\longrightarrow\infty}f_n\big)\mathrm{d}\mu=\int_X\big(\liminf_{n\longrightarrow\infty}f_n\big)\mathrm{d}\mu\le\liminf_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg). \tag{4}$$

For each $n\in\mathbb{Z}^+$, let $h_n=g_n-f_n$. This function is not defined for $x\in X$ such that $f_n(x)=g_n(x)=\infty$; we set $h_n(x)=0$ for such x. Since $g_n\in\mathcal{L}^1(X)$, the set of such x has measure 0 and thus does not affect any statements below. By the first condition in (2), $h_n \geq 0$ a.e. Thus, Fatou's Lemma applies and gives

$$\int_X \big(\liminf_{n\longrightarrow\infty}(g_n - f_n)\big)\mathrm{d}\mu \equiv \int_X \big(\liminf_{n\longrightarrow\infty} h_n\big)\mathrm{d}\mu \leq \liminf_{n\longrightarrow\infty}\bigg(\int_X h_n \mathrm{d}\mu\bigg) = \liminf_{n\longrightarrow\infty}\bigg(\int_X (g_n - f_n)\mathrm{d}\mu\bigg). \tag{5}$$

Using $f_n \longrightarrow f$, $g_n \longrightarrow g$, and $g \in \mathcal{L}^1(X)$, we obtain

$$\int_X \big(\liminf_{n\longrightarrow\infty}(g_n - f_n)\big)\mathrm{d}\mu = \int_X (g - f)\mathrm{d}\mu = \int_X g\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\mu\,. \tag{6}$$

Using $g_n \in \mathcal{L}^1(X)$ and the second condition in (2), we obtain

$$\begin{aligned}\liminf_{n\longrightarrow\infty}\bigg(\int_X (g_n - f_n)\mathrm{d}\mu\bigg) &= \liminf_{n\longrightarrow\infty}\bigg(\int_X g_n\mathrm{d}\mu - \int_X f_n\mathrm{d}\mu\bigg)\\ &= \liminf_{n\longrightarrow\infty}\bigg(\int_X g_n\mathrm{d}\mu\bigg) - \limsup_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg)\\ &= \int_X g\,\mathrm{d}\mu - \limsup_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg).\end{aligned} \tag{7}$$

Combining (5)-(7), we find that

$$\int_X g\,\mathrm{d}\mu - \int_X f\mathrm{d}\mu \leq \int_X g\,\mathrm{d}\mu - \limsup_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg).$$

Since $g \in \mathcal{L}^1(X)$, this gives

$$\int_X f\mathrm{d}\mu \geq \limsup_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg).$$

Combining this with (4), we obtain

$$\limsup_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg) \leq \int_X f\mathrm{d}\mu \leq \liminf_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg) \leq \limsup_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg).$$

Thus, all inequalities above are equalities, which establishes (3) if $f_n \geq 0$ for all n.

In the general case, let $h_n = f_n + g_n$. This function is not defined for $x \in X$ such that $f_n(x) = -\infty$ and $g_n(x) = \infty$ (at the same time); we set $h_n(x) = 0$ for such x. Since $g_n \in \mathcal{L}^1(X)$, the set of such x has measure 0 and thus does not affect any statements below. Since $f_n \longrightarrow f$, $g_n \longrightarrow g$, and $g_n, g \in \mathcal{L}^1(X)$, it follows that

$$h_n \longrightarrow f + g \text{ a.e.}, \qquad 2g_n \longrightarrow 2g \text{ a.e.}, \qquad 2g_n, 2g \in \mathcal{L}^1(X).$$

By (2),

$$0 \leq h_n \leq 2g_n \text{ a.e.} \qquad \text{and} \qquad \int_X (2g)\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\bigg(\int_X (2g_n)\mathrm{d}\mu\bigg).$$

From the conclusion in the previous paragraph, we thus obtain

$$\int_X (f + g)\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\bigg(\int_X h_n\mathrm{d}\mu\bigg) = \lim_{n\longrightarrow\infty}\bigg(\int_X (f_n + g_n)\mathrm{d}\mu\bigg). \tag{8}$$

Since $g_n, g \in \mathcal{L}^1(X)$,

$$\int_X (f+g)\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu,$$

$$\begin{aligned}\lim_{n\longrightarrow\infty}\bigg(\int_X (f_n+g_n)\mathrm{d}\mu\bigg) = \lim_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu + \int_X g_n\mathrm{d}\mu\bigg) &= \lim_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg) + \lim_{n\longrightarrow\infty}\bigg(\int_X g_n\mathrm{d}\mu\bigg)\\ &= \lim_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg) + \int_X g\,\mathrm{d}\mu.\end{aligned}$$

Combining these two equations with (8), we obtain

$$\int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\bigg(\int_X f_n\mathrm{d}\mu\bigg) + \int_X g\,\mathrm{d}\mu\,.$$

Since $g \in \mathcal{L}^1(X)$, this establishes (3).

Problem 3 (10pts)

For each $n \in \mathbb{Z}^+$, define

$$f_n, g_n\colon [0,\infty) \longrightarrow \mathbb{R}, \qquad f_n(x) = \frac{n^2 x\mathrm{e}^{-n^2x^2}}{1+x}, \quad g_n(x) = \frac{x\mathrm{e}^{-x^2}}{1+x/n}\,.$$

(a) *Find* $\displaystyle\int_0^\infty \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}x$ *and* $\displaystyle\int_0^\infty \big(\lim_{n\longrightarrow\infty} g_n\big)\mathrm{d}x$.

(b) *Show that*

$$\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty f_n\mathrm{d}x\bigg) = \lim_{n\longrightarrow\infty}\bigg(\int_0^\infty g_n\mathrm{d}x\bigg)$$

and find this limit.

(c) *Show that there exists no Lebesgue integrable function $F\colon [0,\infty) \longrightarrow [0,\infty]$ such that $f_n \le F$ a.e. on $[0,\infty]$ for all $n \in \mathbb{Z}^+$.*

(a; **3pts**) Since $f_n(0)=0$ for all n, $f_n(0)\longrightarrow 0$. Since $\mathrm{e}^{n^2x^2}$ with $x>0$ dominates every polynomial in n as $n\longrightarrow\infty$, $f_n(x)\longrightarrow 0$ for all $x>0$ as well. It is immediate that

$$g_n(x) \longrightarrow \frac{x\mathrm{e}^{-x^2}}{1+0} = x\mathrm{e}^{-x^2} \qquad \text{as } n\longrightarrow\infty.$$

Thus,

$$\int_0^\infty \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}x = \int_0^\infty 0\mathrm{d}x = 0, \quad \int_0^\infty \big(\lim_{n\longrightarrow\infty} g_n\big)\mathrm{d}x = \int_0^\infty x\mathrm{e}^{-x^2}\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x^2}\Big|_0^\infty = \frac{1}{2}.$$

(b; **4pts**) By the change of variables $x \longrightarrow nx$,

$$\int_0^\infty f_n \mathrm{d}x = \int_0^\infty \frac{(nx)\mathrm{e}^{-(nx)^2}}{1+(nx)/n} \mathrm{d}(nx) = \int_{n\cdot 0}^{n\cdot\infty} \frac{x\mathrm{e}^{-x^2}}{1+x/n} \mathrm{d}x = \int_0^\infty g_n \mathrm{d}x\,.$$

This implies that the two limits in the statement are the same. Since $g_n(x) \ge 0$ and $g_n(x) \nearrow x\mathrm{e}^{-x^2}$ for all $x \in [0,\infty)$,

$$\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty g_n \mathrm{d}x\bigg) = \lim_{n\longrightarrow\infty}\bigg(\int_{[0,\infty)} g_n \mathrm{d}m\bigg) = \int_{[0,\infty)}\big(\lim_{n\longrightarrow\infty} g_n\big)\mathrm{d}m = \int_{[0,\infty)} x\mathrm{e}^{-x^2}\mathrm{d}m = \int_0^\infty x\mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2};$$

the second equality above holds by the Monotone Convergence Theorem.

(c; **3pts**) Suppose such F exists. Since $f_n \ge 0$, the assumption implies that $|f_n| \le F$ for all $n \in \mathbb{Z}^+$. By the Dominated Convergence Theorem and part (a), we would then have

$$\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty f_n \mathrm{d}x\bigg) = \int_0^\infty \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}x = 0.$$

However, this contradicts part (b).

Problem 4 (15pts)

Show that the function

$$f\colon [0,\infty) \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \in \mathbb{R}^+; \\ 0, & \text{if } x = 1; \end{cases}$$

has an improper Riemann integral over $[0,\infty)$*, but is not Lebesgue integrable on* $[0,\infty)$*.*

The function f has an improper Riemann integral over $[0,\infty)$ if the limit

$$\int_0^\infty f(x)\mathrm{d}x \equiv \lim_{a\longrightarrow\infty}\int_0^a f(x)\mathrm{d}x$$

exists. It is Lebesgue integrable on $[0,\infty)$ if the limits

$$\int_{[0,\infty)} f_\pm \mathrm{d}m = \lim_{n\longrightarrow\infty}\int_{[0,n]} f_\pm \mathrm{d}m = \lim_{n\longrightarrow\infty}\int_0^n f_\pm \mathrm{d}x$$

exist (and are finite), where $f_\pm\colon [0,\infty) \longrightarrow [0,\infty)$ are given by

$$f_+(x) = \begin{cases} \frac{|\sin x|}{x}, & \text{if } x \in [\pi(n-1), \pi n] \text{ for some } n \in \mathbb{Z}^+ - 2\mathbb{Z}^+; \\ 0, & \text{otherwise;} \end{cases}$$

$$f_-(x) = \begin{cases} \frac{|\sin x|}{x}, & \text{if } x \in [\pi(n-1), \pi n] \text{ for some } n \in 2\mathbb{Z}^+; \\ 0, & \text{otherwise.} \end{cases}$$

For each $n\in\mathbb{Z}^+$, let

$$a_n = \int_{\pi(n-1)}^{\pi n} \frac{|\sin x|}{x}\mathrm{d}x \geq 0.$$

Since $|\sin(x)|\geq 1/2$ if $x\in[\pi(n-1)+\pi/6, \pi n-\pi/6]$,

$$a_n \geq \frac{1}{2\cdot\pi n}\cdot\frac{2\pi}{3} = \frac{1}{3n}\,.$$

Thus,

$$\lim_{n\longrightarrow\infty}\int_{[0,n]} f_+\mathrm{d}m \geq \sum_{n=1}^{\infty}\frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

because the harmonic series diverges (by the Integral Test for the infinite series). We conclude f is not Lebesgue integrable.

Since $|\sin(x)|\leq 1$,

$$a_n \leq \int_{\pi(n-1)}^{\pi n}\frac{1}{x}\mathrm{d}x \leq \frac{1}{n-1} \longrightarrow 0 \qquad \text{as} \quad n\longrightarrow 0.$$

Since $|\sin(x+\pi)|=|\sin(x)|$,

$$a_n = \int_{\pi(n-1)}^{\pi n}\frac{|\sin(x+\pi)|}{x}\mathrm{d}x \geq \int_{\pi(n-1)}^{\pi n}\frac{|\sin(x+\pi)|}{x+\pi}\mathrm{d}x = \int_{\pi n}^{\pi(n+1)}\frac{|\sin(x+\pi)|}{x}\mathrm{d}x = a_{n+1}\,.$$

By the Alternating Series, the infinite series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n \equiv \lim_{k\longrightarrow\infty}\sum_{n=1}^{k}(-1)^{n-1}a_n = \lim_{k\longrightarrow\infty}\int_0^{\pi k}\frac{\sin x}{x}\mathrm{d}x$$

thus converges. Since

$$\begin{aligned}\int_0^{\pi(k-1)}\frac{\sin x}{x}\mathrm{d}x \leq \int_0^{a}\frac{\sin x}{x}\mathrm{d}x \leq \int_0^{\pi k}\frac{\sin x}{x}\mathrm{d}x \qquad & \text{if } a\in[\pi(k-1),\pi k],\ k\in\mathbb{Z}^+-2\mathbb{Z}^+,\\ \int_0^{\pi k}\frac{\sin x}{x}\mathrm{d}x \leq \int_0^{a}\frac{\sin x}{x}\mathrm{d}x \leq \int_0^{\pi(k-1)}\frac{\sin x}{x}\mathrm{d}x \qquad & \text{if } a\in[\pi(k-1),\pi k],\ k\in 2\mathbb{Z}^+\,,\end{aligned}$$

it follows that

$$\int_0^{\infty} f(x)\mathrm{d}x \equiv \lim_{a\longrightarrow\infty}\int_0^{a} f(x)\mathrm{d}x = \lim_{k\longrightarrow\infty}\sum_{n=1}^{k}(-1)^{n-1}a_n$$

exists. Thus, f has an improper Riemann integral over $[0,\infty)$.