MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 5

Problem 1 (5pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $f_n : X \longrightarrow [0,\infty]$ be a sequence of measurable functions. Show that

$$\int_X \Big(\sum_{n=1}^{\infty} f_n\Big) \mathrm{d}\mu = \sum_{n=1}^{\infty} \Big(\int_X f_n \mathrm{d}\mu\Big).$$

The sequence $g_k \equiv \sum_{n=1}^{k} f_n$ of functions is increasing with k. Each of these functions is measurable, being a finite linear of combination of measurable functions. By the Monotone Convergence Theorem,

$$\int_X \Big(\sum_{n=1}^{\infty} f_n\Big) \mathrm{d}\mu \equiv \int_X \Big(\lim_{k\longrightarrow\infty} \sum_{n=1}^{k} f_n\Big) \mathrm{d}\mu = \int_X \big(\lim_{k\longrightarrow\infty} g_k\big) \mathrm{d}\mu = \lim_{k\longrightarrow\infty} \Big(\int_X g_k \mathrm{d}\mu\Big)$$
$$= \lim_{k\longrightarrow\infty} \bigg(\int_X \Big(\sum_{n=1}^{k} f_n\Big) \mathrm{d}\mu\bigg) = \lim_{k\longrightarrow\infty} \sum_{n=1}^{k} \Big(\int_X f_n \mathrm{d}\mu\Big) \equiv \sum_{n=1}^{\infty} \Big(\int_X f_n \mathrm{d}\mu\Big).$$

The Monotone Convergence Theorem is used for the last equality on the first line above. The integral and *finite* sum on the second line can be switched because we have shown that

$$\int_X (f+g)\mathrm{d}\mu = \int_X f + \int_X g$$

for measurable functions $f,g\colon X \longrightarrow \overline{\mathbb{R}}$.

Problem 2 (10pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $f_n : X \longrightarrow [0,\infty]$ be a sequence of measurable functions decreasing almost everywhere to $f\colon X \longrightarrow [0,\infty]$. Suppose $\int_X f_1 \mathrm{d}\mu < \infty$. Show that

$$\int_X f \mathrm{d}\mu = \lim_{n\longrightarrow\infty} \int_X f_n \mathrm{d}\mu\,. \tag{1}$$

Let $X' \subset X$ be the subset of points such that $f_n(x)$ does not converge to $f(x)$, $X_\infty \subset X$ be the subset of points such that $f_1(x)=\infty$, and

$$Y = X - X' - X_\infty\,.$$

By the assumption, $X' \in \mathcal{F}$ and $\mu(\mathcal{F}')=0$. Since f is measurable, $X_\infty \in \mathcal{F}$. Since $f_1 \in \mathcal{F}^1(X)$, $\mu(X_\infty)=0$. Thus, $Y \in \mathcal{F}$ and

$$\int_X f \mathrm{d}\mu = \int_Y f \mathrm{d}\mu + \int_{X'\cup X_\infty} f \mathrm{d}\mu = \int_Y f \mathrm{d}\mu, \quad \int_X f_n \mathrm{d}\mu = \int_Y f_n \mathrm{d}\mu + \int_{X'\cup X_\infty} f_n \mathrm{d}\mu = \int_Y f_n \mathrm{d}\mu \tag{2}$$

because $\mu(X'\cup X_\infty)=0$. Let $g_n=f_1-f_n$ on Y and $g_n=0$ on $X'\cup X_\infty$. These functions are measurable, being differences of measurable functions. The sequence of these functions is increasing and $g_n\ge 0$ because the sequence f_n is decreasing. By the Monotone Convergence Theorem,

$$\int_Y \big(\lim_{n\longrightarrow\infty} g_n\big)\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\bigg(\int_Y g_n \mathrm{d}\mu\bigg). \tag{3}$$

On the other hand,

$$\int_Y \big(\lim_{n\longrightarrow\infty} g_n\big)\mathrm{d}\mu = \int_Y \big(\lim_{n\longrightarrow\infty}(f_1-f_n)\big)\mathrm{d}\mu = \int_Y \big(f_1-\lim_{n\longrightarrow\infty} f\big)\mathrm{d}\mu = \int_Y f_1\mathrm{d}\mu - \int_Y \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}\mu \tag{4}$$

$$\begin{aligned} \lim_{n\longrightarrow\infty}\bigg(\int_Y g_n\mathrm{d}\mu\bigg) = \lim_{n\longrightarrow\infty}\bigg(\int_Y (f_1-f_n)\mathrm{d}\mu\bigg) &= \lim_{n\longrightarrow\infty}\bigg(\int_Y f_1\mathrm{d}\mu - \int_Y f_n\mathrm{d}\mu\bigg) \\ &= \int_Y f_1\mathrm{d}\mu - \lim_{n\longrightarrow\infty}\bigg(\int_Y f_n\mathrm{d}\mu\bigg). \end{aligned} \tag{5}$$

By (3)-(5),

$$\int_Y f_1\mathrm{d}\mu - \int_Y \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}\mu = \int_Y f_1\mathrm{d}\mu - \lim_{n\longrightarrow\infty}\bigg(\int_Y f_n\mathrm{d}\mu\bigg).$$

Since $\displaystyle\int_Y f_1\mathrm{d}\mu=\int_X f_1\mathrm{d}\mu<\infty$, this gives

$$\int_Y f\mathrm{d}\mu \equiv \int_Y \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}\mu = \lim_{n\longrightarrow\infty}\bigg(\int_Y f_n\mathrm{d}\mu\bigg).$$

Along with (2), this implies (1).

Problem 3 (10pts)

(a) *Let $(X,\mathcal{F},\mu)$ be a measure space and $f: X\longrightarrow[0,\infty)$ be a measurable function. For $n\in\mathbb{Z}$, define*

$$E_n = \big\{x\in X: 2^n < f(x)\le 2^{n+1}\big\}.$$

Show that f is integrable on X if and only if $\displaystyle\sum_{n\in\mathbb{Z}} 2^n\mu(E_n)<\infty.$

(b) *Let $a\in\mathbb{R}$. Use (a) to show that the function $f(x)=x^{-a}$ is Lebesgue integrable on $(0,1)$ if and only if $a<1$.*

(a) Since f is measurable, $E_n\in\mathcal{F}$ for every n. By the definition of E_n,

$$2^n < f\big|_{E_n} \le 2^{n+1} \qquad\Longrightarrow\qquad 2^n\mathbb{1}_{E_n}\big|_{E_n} < f\big|_{E_n} \le 2^{n+1}\mathbb{1}_{E_n}\big|_{E_n}.$$

Since $E_n\cap E_{n'}=\emptyset$ for all $n\neq n'$, it follows that

$$0\le \sum_{n\in\mathbb{Z}} 2^n\mathbb{1}_{E_n} \le f \le \sum_{n\in\mathbb{Z}} 2^{n+1}\mathbb{1}_{E_n} = 2\sum_{n\in\mathbb{Z}} 2^n\mathbb{1}_{E_n},$$

$$0\le \int_X\bigg(\sum_{n\in\mathbb{Z}} 2^n\mathbb{1}_{E_n}\bigg)\mathrm{d}\mu \le \int_X f\mathrm{d}\mu \le \int_X\bigg(2\sum_{n\in\mathbb{Z}} 2^n\mathbb{1}_{E_n}\bigg)\mathrm{d}\mu = 2\int_X\bigg(\sum_{n\in\mathbb{Z}} 2^n\mathbb{1}_{E_n}\bigg)\mathrm{d}\mu.$$

By Problem 1,

$$\int_X \bigg(\sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{E_n}\bigg) \mathrm{d}\mu = \sum_{n\in\mathbb{Z}} \int_X 2^n \mathbb{1}_{E_n} \mathrm{d}\mu = \sum_{n\in\mathbb{Z}} 2^n \int_X \mathbb{1}_{E_n} \mathrm{d}\mu = \sum_{n\in\mathbb{Z}} 2^n \mu(E_n).$$

Thus,

$$0 \le \sum_{n\in\mathbb{Z}} 2^n \mu(E_n) \le \int_X f \mathrm{d}\mu \le 2 \sum_{n\in\mathbb{Z}} 2^n \mu(E_n)\,.$$

This establishes the claim.

(b) If $a>0$, then

$$E_n = \big\{x\in(0,1)\colon 2^{-(n+1)/a} \le x < 2^{-n/a}\big\},$$

$$\sum_{n\in\mathbb{Z}} 2^n \mu(E_n) = \sum_{n=0}^{\infty} 2^n \big(2^{-n/a} - 2^{-(n+1)/a}\big) = \big(1-2^{-1/a}\big) \sum_{n=0}^{\infty} 2^{n(1-1/a)}\,.$$

The last sum above is a geometric series. It converges if $a\in(0,1)$ and diverges if $a\ge 1$. By part (a), this implies that x^{-a} is integrable on $(0,1)$ if $a\in(0,1)$ and is not integrable if $a\ge 1$. If $a\le 0$, then $0\le x^{-a} \le x^{-1/2}$ on $(0,1)$ and thus x^{-a} is also integrable.

Problem 4 (5pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $f\colon X \longrightarrow \overline{\mathbb{R}}$ be a measurable function which is integrable on X. Show that $f\ge 0$ almost everywhere on X if and only if $\int_E f \mathrm{d}\mu \ge 0$ for all $E\in\mathcal{F}$.

Let $X_- = \{x\in X\colon f(x)<0\}$. Since f is measurable, $X'\in\mathcal{F}$. If $E\in\mathcal{F}$, then $E\cap X_-, E-X_- \in\mathcal{F}$ and

$$\int_E f \mathrm{d}\mu = \int_{E-X_-} f \mathrm{d}\mu + \int_{E\cap X_-} f \mathrm{d}\mu\,.$$

The middle integral above is nonnegative because $f|_{E-X_-} \ge 0$. If $\mu(X_-)=0$ (i.e. $f\ge 0$ almost everywhere on X), then $\mu(E\cap X')=0$ and the last integral above vanishes. Thus, $\int_E f \mathrm{d}\mu \ge 0$ for every $E\in\mathcal{F}$.

Suppose $\int_E f \mathrm{d}\mu \ge 0$ for every $E\in\mathcal{F}$. We need to show that $\mu(X_-)=0$. In principle, this is the special case of the first statement of Theorem 4.22 obtained by replacing (f,g) by $(0,f)$. Below is a direct proof. For each $n\in\mathbb{Z}^+$, let

$$E_n = \big\{x\in X\colon x\le -1/n\big\}.$$

Since $f(x)\le -1/n$ on E_n,

$$0 \le \int_{E_n} f \mathrm{d}\mu \le \int_{E_n} (-1/n) \mathrm{d}\mu = -\mu(E_n)/n\,.$$

Thus, $\mu(E_n)=0$ for every $n\in\mathbb{Z}^+$ and so

$$\mu(X_-) = \mu\bigg(\bigcup_{n=1}^{\infty} E_n\bigg) = 0.$$

Problem 5 (10pts)

Show that the limit $\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} x^n \mathrm{e}^{-n|x|}\mathrm{d}m$ *exists and find it.*

The function $x\longrightarrow x\mathrm{e}^{-x}$ reaches the maximum at $x=1$ and this maximum is $e^{-1}<1$. Thus,

$$\left|x^n\mathrm{e}^{-n|x|}\right| = \left(|x|\mathrm{e}^{-|x|}\right)^n \le |x|\mathrm{e}^{-|x|}\,.$$

Since

$$\int_{\mathbb{R}} |x|\mathrm{e}^{-|x|}\mathrm{d}m = 2\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x = 2 < \infty\,,$$

the Dominated Convergence Theorem says

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} x^n\mathrm{e}^{-n|x|}\mathrm{d}m = \int_{\mathbb{R}}\Big(\lim_{n\longrightarrow\infty}\big(x^n\mathrm{e}^{-n|x|}\big)\Big)\mathrm{d}m = \int_{\mathbb{R}} 0\mathrm{d}m = 0.$$

Below is a direct proof of this conclusion.

Since $|x|\mathrm{e}^{-|x|}\le \mathrm{e}^{-1}$,

$$\left|x^n\mathrm{e}^{-n|x|}\right| = \left(|x|\mathrm{e}^{-|x|}\right)^n \le \mathrm{e}^{1-n}|x|\mathrm{e}^{-|x|}\,.$$

Thus,

$$\left|\int_{\mathbb{R}} x^n\mathrm{e}^{-n|x|}\mathrm{d}m\right| \le \int_{\mathbb{R}}\left|x^n\mathrm{e}^{-n|x|}\right|\mathrm{d}m \le \int_{\mathbb{R}}\left(\mathrm{e}^{1-n}|x|\mathrm{e}^{-|x|}\right)\mathrm{d}m = \mathrm{e}^{1-n}\int_{\mathbb{R}}\left(|x|\mathrm{e}^{-|x|}\right)\mathrm{d}m = 2\mathrm{e}^{1-n}\,.$$

Thus, $\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} x^n\mathrm{e}^{-n|x|}\mathrm{d}m=0.$