# MAT 324: Real Analysis, Fall 2017
## Solutions to Problem Set 4

### Problem 1 (10pts)

(a) *Write each* $x\in[0,1]$ *as* $x=\sum_{n=1}^{\infty}\frac{a_n(x)}{2^n}$ *with each* $a_n(x)\in\{0,1\}$ *taking the infinite expansion for all* $x\neq 0$. *Show that the function* $a_n\colon[0,1]\longrightarrow\mathbb{R}$ *is measurable.*

(b) *Show that the function*

$$f\colon[0,1]\longrightarrow\mathbb{R}, \qquad f(x)=\sum_{n=1}^{\infty}\frac{2a_n(x)}{3^n},$$

*is measurable, injective, and takes values in the Cantor set* $C$.

(a) For each $n\in\mathbb{Z}^+$, let

$$A_n=\bigcup_{k=1}^{2^{n-1}}\bigg(\frac{2k-1}{2^n},\frac{2k}{2^n}\bigg].$$

The function $a_n\colon[0,1]\longrightarrow\mathbb{R}$ takes value 1 on $A_n$ and 0 on $[0,1]-A_n$. Thus, $a_n=\mathbb{1}_{A_n}$. Since $A_n$ is a finite union of intervals, $A_n\in\mathcal{M}$. Thus, the indicator function $\mathbb{1}_{A_n}$ is measurable.

(b) Since each function $a_n$ is measurable, so is each function

$$f_k(x)\equiv\sum_{n=1}^{k}\frac{2}{3^n}a_n(x), \qquad k\in\mathbb{Z}^+,$$

because it is a finite linear combination of measurable functions. Since $f=\lim_{k\longrightarrow\infty}f_k$, $f$ is measurable as well.

If $x,x'\in[0,1]$ and $x<x'$, let

$$m=\min\big\{n\in\mathbb{Z}^+\colon a_n(x)\neq a_{n'}(x)\big\}<\infty.$$

Since $x<x'$, $a_m(x)=0$ and $a_m(x')=1$. Thus,

$$f(x)\le\sum_{n=1}^{m-1}\frac{2a_n(x)}{3^n}+\sum_{n=m+1}^{\infty}\frac{2}{3^n}=\sum_{n=1}^{m-1}\frac{2a'_n(x)}{3^n}+\frac{1}{3^m}<f(x').$$

Thus, the function $f$ is strictly increasing (and in particular injective). It takes values between $f(0)=0$ and $f(1)=1$.

Every point $y\in(0,1]$ can be written uniquely as a *non-terminating* infinite sum

$$y=\sum_{n=1}^{\infty}\frac{b_n(y)}{3^n}, \qquad b_n(y)\in\{0,1,2\}.$$

If such $y$ lies in the image of $f$, then $b_n(y)\neq 1$ for all $n\in\mathbb{Z}^+$. Since the Cantor set $C$ consists of the points $y\in(0,1]$ satisfying this condition along with the point 0, it follows that the image of $f$ lies in $C$.

**Problem 2 (17pts)**

(a) *Let* $f\colon X\longrightarrow Y$ *be any map,* $\mathcal{S}\subset 2^Y$*, and* $\mathcal{F}_Y\subset 2^Y$ *be the* $\sigma$*-field generated by* $\mathcal{S}$ *(i.e. the smallest* $\sigma$*-field on* $Y$ *containing* $\mathcal{S}$*). Show that*

$$\mathcal{F}_X \equiv \big\{f^{-1}(B)\colon B\in\mathcal{F}_Y\big\}$$

*is the* $\sigma$*-field generated by* $\{f^{-1}(B)\colon B\in\mathcal{S}\}$*.*

(b) *Let* $(X,\mathcal{F},\mu)$ *be a measure space and* $f\colon X\longrightarrow\mathbb{R}$ *be a measurable function. Show that* $f^{-1}(B)\in\mathcal{F}$ *for every Borel subset* $B\subset\mathbb{R}$ *(i.e.* $B\in\mathcal{B}$*).*

(c) *Give an example of a measurable function* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *and a measurable subset* $E\subset\mathbb{R}$ *(i.e.* $E\in\mathcal{M}$*) so that* $f^{-1}(E)$ *is not measurable.*

(d) *Give an example of measurable functions* $f,g\colon\mathbb{R}\longrightarrow\mathbb{R}$ *so that* $f\circ g\colon\mathbb{R}\longrightarrow\mathbb{R}$ *is not measurable.*

(a) Since $\mathcal{S}\subset\mathcal{F}_Y$,

$$f^{-1}(\mathcal{S})\equiv\big\{f^{-1}(B)\colon B\in\mathcal{S}\big\}\subset\mathcal{F}_X\,.$$

Since $Y\in\mathcal{F}_Y$ (because $\mathcal{F}_Y$ is a $\sigma$-field on $Y$), $X=f^{-1}(Y)\in\mathcal{F}_X$. If $A\in\mathcal{F}_X$, then $A=f^{-1}(B)$ for some $B\in\mathcal{F}_Y$ and

$$X-A = X-f^{-1}(B) = f^{-1}(Y-B)\in\mathcal{F}_X$$

because $Y–B\in\mathcal{F}_Y$. If $A_1,A_2,\ldots\in\mathcal{F}_X$, then $A_1=f^{-1}(B_1), A_2=f^{-1}(B_2),\ldots$ for some $B_1,B_2,\ldots\in\mathcal{F}_Y$ and

$$\bigcup_{n=1}^{\infty}A_n = \bigcup_{n=1}^{\infty}f^{-1}(B_n) = f^{-1}\bigg(\bigcup_{n=1}^{\infty}B_n\bigg)\in\mathcal{F}_X$$

because $\bigcup B_n\in\mathcal{F}_Y$. Thus, $\mathcal{F}_X$ is a $\sigma$-field on $X$ containing $f^{-1}(\mathcal{S})$.

Suppose $\mathcal{F}_X'$ is any $\sigma$-field on $X$ containing $f^{-1}(\mathcal{S})$. Let

$$\mathcal{F}_Y' = \big\{B\in\mathcal{F}_Y\colon f^{-1}(B)\in\mathcal{F}_X'\big\}.$$

Since $f^{-1}(\mathcal{S})\subset\mathcal{F}_X'$, $\mathcal{S}\subset\mathcal{F}_Y'$. Since $X=f^{-1}(Y)\in\mathcal{F}_X'$ (because $\mathcal{F}_X'$ is a $\sigma$-field on $X$), $Y\in\mathcal{F}_X'$. If $B\in\mathcal{F}_Y'$, then $f^{-1}(B)\in\mathcal{F}_X'$, $Y-B\in\mathcal{F}_Y$, and

$$f^{-1}(Y-B) = X-f^{-1}(B)\in\mathcal{F}_X'\,;$$

thus, $Y-B\in\mathcal{F}_Y'$. If $B_1,B_2,\ldots\in\mathcal{F}_Y'$, then $f^{-1}(B_1),f^{-1}(B_2),\ldots\in\mathcal{F}_X'$, $\bigcup B_n\in\mathcal{F}_Y$, and

$$f^{-1}\bigg(\bigcup_{n=1}^{\infty}B_n\bigg) = \bigcup_{n=1}^{\infty}f^{-1}(B_n)\in\mathcal{F}_X';$$

thus, $\bigcup B_n\in\mathcal{F}_Y'$. We conclude that $\mathcal{F}_Y'\subset\mathcal{F}_Y$ is a $\sigma$-field on $Y$ containing $\mathcal{S}$ and thus $\mathcal{F}_Y'=\mathcal{F}_Y$. Therefore, $\mathcal{F}_X\subset\mathcal{F}_X'$ and so $\mathcal{F}_X$ is the smallest $\sigma$-field on $X$ containing $\mathcal{S}$.

(b) Since $\mathcal{B}$ is the $\sigma$-field on $\mathbb{R}$ generated by the collection $\mathcal{S}$ of the intervals $I\subset\mathbb{R}$, part (a) implies that

$$f^{-1}(\mathcal{B})\equiv\big\{f^{-1}(B)\colon B\in\mathcal{B}\big\}\subset 2^X$$

is the $\sigma$-field on $X$ generated by the set

$$f^{-1}(\mathcal{S}) \equiv \{f^{-1}(I) : I \in \mathcal{S}\}.$$

Since $f$ is a measurable function, $f^{-1}(\mathcal{S}) \subset \mathcal{F}$. Since $\mathcal{F}$ is a $\sigma$-field on $X$, it follows that $f^{-1}(\mathcal{B}) \subset \mathcal{F}$.

(c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined on $[0,1]$ as in Problem 1(b) and given by $f(x) = x$ for $x \notin [0,1]$. This function is measurable on $\mathbb{R} - [0,1]$ because it agrees with the continuous function $f(x) = x$ there and is measurable on $[0,1]$ by Problem 1(b). It is also injective because it is injective on $\mathbb{R}$–$[0,1]$, takes values in $\mathbb{R}$–$[0,1]$ there, and is injective on $[0,1]$, and takes values in $[0,1]$ there.

Let $B \subset [0,1]$ be a non-measurable subset, e.g. as constructed on p302, and $E = f(B)$. Since $E$ is contained in the Cantor set $C$, it is a null set and thus $E \in \mathcal{M}$. Since $f$ is injective, $f^{-1}(E) = B$ is a non-measurable set.

(d) Let $f$ and $E$ be as in (c) and $g = \mathbb{1}_E : \mathbb{R} \longrightarrow \mathbb{R}$. Since $E$ is a measurable set, its indicator function $g$ is measurable. However, the function

$$h \equiv g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$$

is not measurable because

$$h^{-1}(1) = f^{-1}\big(g^{-1}(1)\big) = f^{-1}(E)$$

is not a measurable set.

**Problem 3 (15pts)**

*Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Show that the set*

$$E'_f \equiv \{x \in \mathbb{R} : f \text{ is differentiable at } x\}$$

*is measurable.*

For $r \in \mathbb{R}^+$, we let $\mathbb{Q}_r^* \subset \mathbb{Q}$ denote the subset of rational numbers $q \neq 0$ such that $|q| < r$. Recall that

$$f'(x) = \lim_{y \longrightarrow 0} \frac{f(x+y) - f(x)}{y}$$

if this limit exists (and is finite); the limit here is taken over $y \in \mathbb{R}^* \equiv \mathbb{R} - \{0\}$. We first note that it is sufficient to take this limit over $y \in \mathbb{Q}^* \equiv \mathbb{Q} - \{0\}$, i.e.

$$E'_f = \left\{ x \in \mathbb{R} : \lim_{\substack{q \in \mathbb{Q}^* \\ q \longrightarrow 0}} \frac{f(x+q) - f(x)}{q} \text{ exists (and is finite)} \right\}. \tag{1}$$

Suppose the last limit exists for some $x$ and equals $c \in \mathbb{R}$. For every $\varepsilon > 0$, there then exists $\delta > 0$ such that

$$\left| \frac{f(x+q) - f(x)}{q} - c \right| < \frac{\varepsilon}{2} \qquad \forall\, q \in \mathbb{Q}_\delta^*.$$

Since the function

$$\mathbb{R}^* \longrightarrow \mathbb{R}, \qquad y \longrightarrow \frac{f(x+y) - f(x)}{y} - c,$$

is continuous, for every $y\in\mathbb{R}^*$ there exists $\delta_y\in(0,|y|)$ such that

$$\left|\left(\frac{f(x+q)-f(x)}{q}-c\right)-\left(\frac{f(x+y)-f(x)}{y}-c\right)\right|<\frac{\varepsilon}{2}\qquad\forall\, q\in\mathbb{Q}^* \text{ s.t. } |y-q|<\delta_y.$$

Combining the two bounds above, we find that

$$\left|\frac{f(x+y)-f(x)}{y}-c\right|<\varepsilon\qquad\forall\, y\in\mathbb{R}^* \text{ s.t. } |y|<\delta\,.$$

This establishes (1).

For each $n\in\mathbb{Z}^+$, define

$$g_n\colon\mathbb{R}\longrightarrow\mathbb{R},\qquad g_n(x)=\big(f(x+1/n)-f(x)\big)/(1/n)=n\big(f(x+1/n)-f(x)\big).$$

Each of these functions is measurable, being a finite linear combination of measurable functions. The function

$$g\equiv\inf_{n\in\mathbb{Z}^+}g_n\colon\mathbb{R}\longrightarrow[-\infty,\infty),\qquad g(x)=\inf\big\{g_n(x)\colon n\in\mathbb{Z}^+\big\},$$

is thus also measurable. In particular, the set $E\equiv g^{-1}(\mathbb{R})$ is measurable.

If the derivative $f'(x)$ of $f$ at $x$ exists (and is finite), then $x\in E$ and $f'(x)=g(x)$. However, even if $x\in E$, $f'(x)$ need not exist. For each $q\in\mathbb{Q}^*$, define

$$h_q\colon\mathbb{R}\longrightarrow[0,\infty],\qquad h_q(x)=\left|\frac{f(x+q)-f(x)}{q}-g(x)\right|.$$

Since $h_q$ is the absolute value of a finite linear combination of measurable functions, it is measurable. Let

$$F=\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{q\in\mathbb{Q}^*_{1/m}}\left\{x\in\mathbb{R}: g_q(x)<\frac{1}{n}\right\}.$$

Since $F$ is a countable intersection of countable unions of countable intersections of measurable subsets of $\mathbb{R}$, it is measurable. By (1), $E'_f=E\cap F$. Thus, $E'_f$ is measurable.

**Problem 4 (8pts)**

*A car leaves point A at random between 1pm and 2pm and travels at 50mph towards point B, which is 20 miles away. Find the probability distribution of the distance at 1:50pm.*

Let $\omega\in[0,1]$ denote the starting time measured as a fraction of an hour from 1pm and $X(\omega)\in[0,20]$ denote the distance from $B$ at 1:50, i.e. at the time 5/6 in these units. Since it takes 2/5 of a unit to get from $A$ to $B$,

$$X(\omega)=\begin{cases}0, & \text{if } \omega\in[0,13/30];\\ 50\omega-\frac{65}{3} & \text{if } \omega\in(13/30,5/6);\\ 20, & \text{if } \omega\in[5/6,1].\end{cases}$$

If $B\subset\mathbb{R}$, then

$$\begin{aligned}X^{-1}(B)&=\big\{\omega\in[0,13/30]: 0\in B\big\}\cup\Big\{\omega\in(13/30,5/6): 50\omega-\frac{65}{3}\in B\Big\}\cup\big\{\omega\in[5/6,1]: 20\in B\big\}\\&=\frac{1}{50}\big((0,20)\cap B+65/3\big)\cup\begin{cases}[0,13/30], & \text{if } 0\in B;\\ \emptyset, & \text{if } 0\notin B;\end{cases}\cup\begin{cases}[5/6,1], & \text{if } 20\in B;\\ \emptyset, & \text{if } 20\notin B.\end{cases}\end{aligned}$$

Thus, if $B$ is Borel (or even measurable),

$$P_X(B) \equiv m\big(X^{-1}(B)\big) = \frac{1}{50}m\big([0,20]\cap B\big) + \begin{cases} 13/30, & \text{if } 0\in B; \\ \emptyset, & \text{if } 0\notin B; \end{cases} + \begin{cases} 1/6, & \text{if } 20\in B; \\ \emptyset, & \text{if } 20\notin B. \end{cases}$$

In other words, $P_X = \frac{13}{30}\delta_0 + \frac{1}{6}\delta_{20} + \frac{1}{50}m_{[0,20]} : \mathbb{R} \longrightarrow [0,1]$.

**Problem 5 (10pts)**

*Let $F: [0,1] \longrightarrow [0,1]$ be the Lebesgue function defined at the top of p20. Find* $\int_{[0,1]} F\mathrm{d}m$.

For each $n\in\mathbb{Z}^{\geq 0}$, let

$$S_n = \bigg\{ \sum_{k=1}^{n} \frac{a_k}{3^k} : a_k \in \{0,2\} \bigg\}$$

be the set of the left endpoints of the $2^n$ intervals making up the set $C_n$ on page 19. For each $n\in\mathbb{Z}^+$, define

$$A_n = \bigcup_{a\in S_{n-1}} \bigg(a + \frac{1}{3^n}, \frac{3}{3^n}\bigg], \qquad \varphi_n = \sum_{\ell=1}^{n} \frac{1}{2^\ell}\mathbb{1}_{A_\ell} : [0,1] \longrightarrow [0,\infty)\,.$$

Since $\varphi_n$ is a finite sum of step functions as in Definition 4.1, it is also a step function as in Definition 4.1 and

$$\int_{[0,1]} \varphi_n \mathrm{d}m = \sum_{\ell=1}^{n} \frac{1}{2^\ell} m(A_\ell) = \sum_{\ell=1}^{n} \frac{1}{2^\ell} 2^{\ell-1} \frac{2}{3^\ell} = \frac{1}{2}\big(1 - 3^{-n}\big).$$

Furthermore, $\varphi_1 \leq \varphi_2$ and $\varphi_n(x) \longrightarrow F(x)$ as $n \longrightarrow \infty$ for all $x\in[0,1]$. By the Monotone Convergence Theorem, this implies that

$$\int_{[0,1]} F\mathrm{d}m = \lim_{n\longrightarrow\infty} \int_{[0,1]} \varphi_n \mathrm{d}m = \lim_{n\longrightarrow\infty} \frac{1}{2}\big(1 - 3^{-n}\big) = \frac{1}{2}.$$

However, this is later in the book. Below is a direct argument.

Since $\varphi_n \leq F$,

$$\int_{[0,1]} F\mathrm{d}m \geq \int_{[0,1]} \varphi_n \mathrm{d}m = \frac{1}{2}\big(1 - 3^{-n}\big) \qquad \forall\, n\in\mathbb{Z}^+.$$

Since

$$F(x) - \varphi_n(x) \leq \sum_{\ell=n+1}^{\infty} \frac{1}{2^\ell} = \frac{1}{2^n} \qquad \forall\, x\in[0,\infty],\, n\in\mathbb{Z}^+,$$

we find that

$$\int_{[0,1]} F\mathrm{d}m \leq \int_{[0,1]} \big(\varphi_n + 2^{-n}\big)\mathrm{d}m = \frac{1}{2}\big(1 - 3^{-n}\big) + 2^{-n}\,.$$

Combining this with the above estimate, we obtain

$$\frac{1}{2} - \frac{1}{2}3^{-n} \leq \int_{[0,1]} F\mathrm{d}m \leq \frac{1}{2} - \frac{1}{2}3^{-n} + 2^{-n} \qquad \forall\, n\in\mathbb{Z}^+\,.$$

This implies that $\int_{[0,1]} F\mathrm{d}m = \frac{1}{2}$.