MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 10

Problem 1 (16pts)

Suppose X is a set and $\mu^*: 2^X \longrightarrow [0,\infty]$ is a function such that

$$\mu^*(\emptyset)=0, \quad \mu^*(A)\le\mu^*(B) \text{ if } A\subset B, \quad \text{and} \quad \mu^*\bigg(\bigcup_{n=1}^{\infty}A_n\bigg) \le \sum_{n=1}^{\infty}\mu^*(A_n) \;\; \forall\, A_1,A_2,\ldots\subset X. \tag{1}$$

Define

$$\mathcal{M}_{\mu^*} = \big\{E\subset X: \mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^c)\ \forall\, A\subset X\big\}.$$

Show that

(a) $\mu^*(A\cup B)\le\mu^*(A)+\mu^*(B)$ *for all* $A,B\subset X$;

(b) $X\in\mathcal{M}_{\mu^*}$, $E^c\in\mathcal{M}_{\mu^*}$ *if* $E\in\mathcal{M}_{\mu^*}$, *and* $E\in\mathcal{M}_{\mu^*}$ *if* $\mu^*(E)=0$;

(c) *if* $E,F\in\mathcal{M}_{\mu^*}$ *and* $A\subset X$, *then*

$$\begin{aligned}\mu^*(A) &= \mu^*\big(A\cap E\cap F\big)+\mu^*\big(A\cap E\cap F^c\big)+\mu^*\big(A\cap E^c\cap F\big)+\mu^*\big(A\cap E^c\cap F^c\big)\\ &\ge \mu^*\big(A\cap(E\cup F)\big)+\mu^*\big(A\cap(E\cup F)^c\big);\end{aligned}$$

(d) $\mathcal{M}_{\mu^*}$ *is a field (not σ-field yet) on X and*

$$\mu^*\bigg(\bigcup_{n=1}^{k}E_n\bigg) = \sum_{n=1}^{k}\mu^*(E_n) \quad \forall\ E_1,\ldots,E_k\in\mathcal{M}_{\mu^*} \text{ s.t. } E_n\cap E_{n'}=\emptyset\ \forall\, n\neq n'; \tag{2}$$

(e) *if* $E_1,E_2,\ldots\in\mathcal{M}_{\mu^*}$ *and* $E_n\cap E_{n'}=\emptyset$ *for all* $n\neq n'$, *then*

$$\mu^*\bigg(A\cap\bigcup_{n=1}^{\infty}E_n\bigg) = \sum_{n=1}^{\infty}\mu^*(A\cap E_n) \;\; \forall\ A\subset X, \quad \mu^*\bigg(\bigcup_{n=1}^{\infty}E_n\bigg) = \sum_{n=1}^{\infty}\mu^*(E_n); \tag{3}$$

(f) $(X,\mathcal{M}_{\mu^*},\mu\equiv\mu^*|_{\mathcal{M}_{\mu^*}})$ *is a complete measure space.*

(a; **2pts**) By the third condition in (1) with $A_n=\emptyset$ for $n>k$ and the first condition in (1),

$$\mu^*\bigg(\bigcup_{n=1}^{k}A_n\bigg) \le \sum_{n=1}^{k}\mu^*(A_n) \qquad \forall\, A_1,A_2,\ldots,A_k\subset X,\ k\in\mathbb{Z}^{\ge0}. \tag{4}$$

The claim is the $k=2$ case of this statement with $A_1=A$ and $A_2=B$.

(b; **2pts**) By (a) with (A,B) replaced by $(A\cap E, A\cap E^c)$,

$$\mu^*(A) \le \mu^*(A\cap E)+\mu^*(A\cap E^c) \qquad \forall\ A,E\subset X.$$

Thus,

$$\mathcal{M}_{\mu^*} = \big\{E\subset X : \mu^*(A\cap E)+\mu^*(A\cap E^c)\le\mu^*(A)\ \forall\, A\subset X\big\}. \tag{5}$$

Let $A\subset X$ be arbitrary. Since $A\cap X=A$ and $A\cap X^c=\emptyset$, $X\in\mathcal{M}_{\mu^*}$. If $E\in\mathcal{M}_{\mu^*}$, then

$$\mu^*(A\cap E^c)+\mu^*\big(A\cap(E^c)^c\big) = \mu^*(A\cap E^c)+\mu^*(A\cap E) \le \mu^*(A)$$

and so $E^c\in\mathcal{M}_{\mu^*}$. If $\mu^*(E)=0$, then the second condition in (1) gives

$$\mu^*(A\cap E)+\mu^*(A\cap E^c) \le \mu^*(E)+\mu^*(A) = \mu^*(A).$$

Thus, $E\in\mathcal{M}_{\mu^*}$.

(c; **2pts**) If $E,F\in\mathcal{M}_{\mu^*}$ and $A\subset X$,

$$\begin{aligned}\mu^*(A) &= \mu^*(A\cap E)+\mu^*(A\cap E^c)\\ &= \big(\mu^*\big((A\cap E)\cap F\big)+\mu^*\big((A\cap E)\cap F^c\big)\big) + \big(\mu^*\big((A\cap E^c)\cap F\big)+\mu^*\big((A\cap E^c)\cap F^c\big)\big);\end{aligned}$$

this establishes the equality in the claim. Since

$$E\cup F = (E\cap F)\cup(E\cap F^c)\cup(E^c\cap F),$$

the $k=3$ case of (4) gives

$$\mu^*\big(A\cap E\cap F\big)+\mu^*\big(A\cap E\cap F^c\big)+\mu^*\big(A\cap E^c\cap F\big) \ge \mu^*\big(A\cap(E\cup F)\big).$$

Along with $E^c\cap F^c=(E\cup F)^c$, this establishes the inequality in the claim.

(d; **3pts**) Let $A\subset X$ be arbitrary. By (c),

$$\mu^*\big(A\cap(E\cup F)\big)+\mu^*\big(A\cap(E\cup F)^c\big) \le \mu^*(A) \qquad \forall\, E,F\in\mathcal{M}_{\mu^*},\ A\subset X.$$

By (5), this implies that $\mathcal{M}_{\mu^*}$ is closed under pairwise (and thus finite) unions. Along with the first two statements in (b), this implies that $\mathcal{M}_{\mu^*}$ is a field. If $E\in\mathcal{M}_{\mu^*}$ and $F\subset X$ is disjoint from E, then

$$\mu^*(F\cup E) = \mu^*\big((F\cup E)\cap E\big)+\mu^*\big((F\cup E)\cap E^c\big) = \mu^*(E)+\mu^*(F).$$

This establishes the $k=2$ case of (2), which by induction implies the general case.

(e; **3pts**) Let $E_1,E_2,\ldots\in\mathcal{M}_{\mu^*}$ with $E_n\cap E_{n'}=\emptyset$ for all $n\neq n'$ and $A\subset X$. If $E\in\mathcal{M}_{\mu^*}$ and $F\subset X$ is disjoint from E, then

$$\mu^*\big(A\cap(E\cup F)\big) = \mu^*\big(A\cap(E\cup F)\cap E\big)+\mu^*\big(A\cap(E\cup F)\cap E^c\big) = \mu^*(A\cap E)+\mu^*\big(A\cap F\big).$$

By the closedness of $\mathcal{M}_{\mu^*}$ under finite unions established in (d) and induction, this implies that

$$\mu^*\bigg(A\cap\bigcup_{n=1}^{k}E_n\bigg) = \sum_{n=1}^{k}\mu^*(A\cap E_n) \qquad \forall\ k\in\mathbb{Z}^{\ge0}. \tag{6}$$

Combining this with the second condition in (1), we obtain

$$\sum_{n=1}^{\infty}\mu^*(A\cap E_n)=\lim_{k\longrightarrow\infty}\sum_{n=1}^{k}\mu^*(A\cap E_n)=\lim_{k\longrightarrow\infty}\mu^*\biggl(A\cap\bigcup_{n=1}^{k}E_n\biggr)\le\mu^*\biggl(A\cap\bigcup_{n=1}^{\infty}E_n\biggr).$$

Along with the third condition in (1), this implies the first statement in (3). The second statement in (3) is obtained from the first by taking $A=\bigcup_{n=1}^{\infty}E_n$.

(f; **4pts**) Let $E_1,E_2,\ldots\in\mathcal{M}_{\mu^*}$, $A\subset X$, and

$$F_n=E_n-\bigcup_{k=1}^{n-1}E_k \qquad \forall\, n\in\mathbb{Z}^{\ge0}.$$

Thus, $F_n\cap F_{n'}=\emptyset$ for all $n\neq n'$ and $\bigcup_{n=1}^{\infty}E_n=\bigcup_{n=1}^{\infty}F_n$. By (d), $F_n\in\mathcal{M}_{\mu^*}$. Combining this with (6) with E_n replaced by F_n, the second condition in (1), and the closed of $\mathcal{M}_{\mu^*}$ under finite unions, we obtain

$$\begin{aligned}\sum_{n=1}^{k}\mu^*(A\cap F_n)+\mu^*\biggl(A\cap\Bigl(\bigcup_{n=1}^{\infty}E_n\Bigr)^c\biggr)&=\mu^*\biggl(A\cap\bigcup_{n=1}^{k}F_n\biggr)+\mu^*\biggl(A\cap\Bigl(\bigcup_{n=1}^{\infty}F_n\Bigr)^c\biggr)\\&\le\mu^*\biggl(A\cap\bigcup_{n=1}^{k}F_n\biggr)+\mu^*\biggl(A\cap\Bigl(\bigcup_{n=1}^{k}F_n\Bigr)^c\biggr)=\mu^*(A).\end{aligned}$$

Taking the limit of the left-hand side as $k\longrightarrow\infty$ and using the first statement in (3), we find that

$$\mu^*\biggl(A\cap\bigcup_{n=1}^{\infty}E_n\biggr)+\mu^*\biggl(A\cap\Bigl(\bigcup_{n=1}^{\infty}E_n\Bigr)^c\biggr)=\mu^*\biggl(A\cap\bigcup_{n=1}^{\infty}F_n\biggr)+\mu^*\biggl(A\cap\Bigl(\bigcup_{n=1}^{\infty}E_n\Bigr)^c\biggr)\le\mu^*(A).$$

Thus, $\mathcal{M}_{\mu^*}$ is closed under countable unions and so is a σ-field (in light of the first two statements in (b)). By the second statement in (3), $\mu\equiv\mu^*|_{\mathcal{M}_\mu^*}$ is countably additive. Thus, $(X,\mathcal{M}_{\mu^*},\mu\equiv\mu^*|_{\mathcal{M}_{\mu^*}})$ is a measure space. It is complete by the last statement in (b).

Problem 2 (12pts)

Let X be a set, $\mathcal{A}\subset 2^X$, and $\ell\colon\mathcal{A}\longrightarrow[0,\infty]$ be a function such that $\emptyset\in\mathcal{A}$ and $\ell(\emptyset)=0$. For each $A\subset X$, define

$$Z_\ell(A)=\biggl\{\sum_{n=1}^{\infty}\ell(I_n)\colon I_1,I_2,\ldots\in\mathcal{A},\ A\subset\bigcup_{n=1}^{\infty}I_n\biggr\}\subset[0,\infty],\quad \mu_\ell^*(A)=\inf Z_\ell(A)\in[0,\infty].$$

(a) *Show that $\mu^*\equiv\mu_\ell^*$ satisfies (1).*

(b) *Suppose in addition that $\mathcal{A}$ is a field (not necessarily σ-field) on X and*

$$\ell\biggl(\bigcup_{n=1}^{\infty}I_n\biggr)=\sum_{n=1}^{\infty}\ell(I_n)\quad\forall\ I_1,I_2,\ldots\in\mathcal{A}\ \text{ s.t. }\ I_n\cap I_{n'}=\emptyset\ \forall\,n\neq n'\ \text{and}\ \bigcup_{n=1}^{\infty}I_n\in\mathcal{A}. \tag{7}$$

Show that $\mu_\ell^*|_{\mathcal{A}}=\ell$ and $\mathcal{A}\subset\mathcal{M}_{\mu_\ell^*}$.

(c) *What should $\mathcal{A}$ and ℓ be taken to construct the Lebesgue measure m_n on $\mathbb{R}^n$? Justify your answer.*

(a; **3pts**) Since $\emptyset\in\mathcal{A}$ and $\ell(\emptyset)=0$, $0\in Z_\ell(\emptyset)$ and so $\mu_\ell^*(\emptyset)=0$. If $A\subset B$, $Z_\ell(A)\supset Z_\ell(B)$ and so $\mu_\ell^*(A)\le\mu_\ell^*(B)$. For every $\epsilon>0$ and $A_k\subset X$, there exist

$$I_{k;1},I_{k;2},\ldots\in\mathcal{A} \qquad \text{s.t.} \quad A_k\subset\bigcup_{n=1}^{\infty}I_{k;n}, \quad \sum_{n=1}^{\infty}\ell(I_{k;n})\le\mu_\ell^*(A_k)+\frac{\epsilon}{2^k}\,.$$

Thus,

$$\bigcup_{k=1}^{\infty}A_k\subset\bigcup_{k=1}\bigcup_{n=1}^{\infty}I_{k;n}, \quad \mu_\ell^*\bigg(\bigcup_{k=1}^{\infty}A_k\bigg)\le\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\ell(I_{k;n})\le\sum_{n=1}^{\infty}\big(\mu^*(A_n)+\epsilon/2^n\big)=\sum_{n=1}^{\infty}\mu^*(A_n)+\epsilon\,.$$

This verifies the three properties in (1).

(b; **6pts**) By the assumption that $\emptyset\in\mathcal{A}$ and $\ell(\emptyset)=0$, $\ell(I)\in Z_\ell(I)$ for all $I\in\mathcal{A}$ and so

$$m_\ell^*(I)\le\ell(I) \qquad \forall\, I\in\mathcal{A}. \tag{8}$$

Since $\mathcal{A}$ is a field, $A\cup B, B-A\in\mathcal{A}$ whenever $A,B\in\mathcal{A}$. By (7) with $I_1=A$, $I_2=B$, and $I_n=\emptyset$ for $n\ge3$ and by the assumption that $\emptyset\in\mathcal{A}$ and $\ell(\emptyset)=0$,

$$\ell(A\cup B)=\ell(A)+\ell(B) \qquad \forall\, A,B\in\mathcal{A}\text{ s.t. } A\cap B=\emptyset. \tag{9}$$

Applying this with B replaced by $B-A$, we obtain

$$\ell(A)\le\ell(B) \qquad \forall\, A,B\in\mathcal{A}\text{ s.t. } A\subset B. \tag{10}$$

If $I,I_1,I_2,\ldots\in\mathcal{A}$ and $I\subset I_1\cup I_2\cup\ldots$, then

$$I'_n\equiv I\cap I_n-\bigcup_{k=1}^{n-1}I_n\in\mathcal{A}, \quad I=\bigcup_{n=1}^{\infty}I'_n, \quad I'_n\cap I'_{n'}=\emptyset\ \forall\, n\ne n', \quad \ell(I)=\bigcup_{n=1}^{\infty}\ell(I'_n)\le\bigcup_{n=1}^{\infty}\ell(I_n);$$

the equality above holds by (7) and the inequality by (10). By the last statement above, $\ell(I)\le m_\ell^*(I)$. Combining this with (8), we conclude that $\mu_\ell^*|_{\mathcal{A}}=\ell$.

Suppose $I\in\mathcal{A}$ and $A\subset X$. For every $\epsilon>0$, there exist

$$I_1,I_2,\ldots\in\mathcal{A} \quad \text{s.t.} \quad A\subset\bigcup_{n=1}^{\infty}I_n, \quad \sum_{n=1}^{\infty}\ell(I_n)\le\mu_\ell^*(A)+\epsilon.$$

Since $\mathcal{A}$ is a field,

$$I_n\cap I, I_n\cap I^c\in\mathcal{A}, \quad \ell(I)=\ell(I_n\cap I)+\ell(I_n\cap I^c) \qquad \forall\, n\in\mathbb{Z}^+;$$

the equality above holds by (9). Since

$$A\cap I\subset\bigcup_{n=1}^{\infty}(I_n\cap I), \qquad A\cap I^c\subset\bigcup_{n=1}^{\infty}(I_n\cap I^c),$$

we conclude that

$$\begin{aligned}\mu_\ell^*(A\cap I)+\mu_\ell^*(A\cap I^c) \le \sum_{n=1}^{\infty}\ell(I_n\cap I)+\sum_{n=1}^{\infty}\ell(I_n\cap I^c) &= \sum_{n=1}^{\infty}\big(\ell(I_n\cap I)+\ell(I_n\cap I^c)\big)\\ &= \sum_{n=1}^{\infty}\ell(I_n) \le \mu_\ell^*(A)+\epsilon.\end{aligned}$$

Since this inequality holds for $\epsilon>0$, it follows that it also holds for $\epsilon=0$ and so $I\in\mathcal{M}_{\mu_\ell^*}$.

(c; **3pts**) We should take $\mathcal{A}$ to be the collection of finite unions of disjoint boxes $I_1\times\ldots\times I_n$, where each $I_k\subset\mathbb{R}$ is an interval of any kind (open/closed/half-open, possibly infinite or half-infinite). This collection is a field if $n=1$, as can be seen directly. The argument at the beginning of the proof of Theorem 8.3 can then be used to carry out the inductive step (as n increases) in showing that $\mathcal{A}$ is a field. For such a rectangle, we take

$$\ell(I_1\times\ldots\times I_n) = \ell_1(I_1)\ldots\ell_1(I_n),$$

where $\ell_1(\cdot)$ is the usual length of an interval in $\mathbb{R}$ (and $0\cdot\infty\equiv 0$).

It is immediate that $\emptyset\in\mathcal{A}$ and $\ell(\emptyset)=0$. The tricky part is to verify (7). It is enough to establish it when

$$I \equiv \bigcup_{n=1}^{\infty} I_n \in \mathcal{A}$$

is a single rectangle, instead of a finite union of disjoint rectangles, and is bounded and closed (because an arbitrary rectangle differs from a closed one by missing finitely many disjoint rectangle of n-dimensional length 0). Since the rectangles I_n are disjoint and are contained in I,

$$\sum_{n=1}^{k}\ell(I_n) \le \ell(I) \quad \forall\, k\in\mathbb{Z}^+ \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty}\ell(I_n) \le \ell(I)\,.$$

It remains to establish the opposite inequality. Let $\epsilon>0$. For each $n\in\mathbb{Z}^+$, let I_n' be an open rectangle such that

$$I_n \subset I_n' \qquad \text{and} \qquad \ell(I_n') \le \ell(I_n) + \frac{\epsilon}{2^n}\,.$$

Since $\{I_n\}_{n\in\mathbb{Z}^+}$ is a collection of open sets covering the compact set I, there exists $k\in\mathbb{Z}^+$ such that

$$I \subset \bigcup_{n=1}^{k} I_n' \qquad \text{and} \qquad \ell(I) \le \sum_{n=1}^{k}\ell(I_n') \le \sum_{n=1}^{\infty}\ell(I_n)+\epsilon;$$

the finiteness of the cover is used to obtain the first inequality above. This establishes the opposite inequality.

Problem 3 (12pts)

Let $(X,\mathcal{F}_1,\mu_1)$ and $(Y,\mathcal{F}_2,\mu_2)$ be σ-finite measure spaces and $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$ be the σ-field generated by

$$\mathcal{F}_1\times\mathcal{F}_2 \equiv \{A\times B\colon A\in\mathcal{F}_1,\ A\in\mathcal{F}_2\}.$$

Suppose that μ is a measure on $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$ such that

$$\mu(A\times B)=\mu_1(A)\mu_2(B) \qquad \forall\ A\times B\in\mathcal{F}_1\times\mathcal{F}_2. \tag{11}$$

Show that μ is the product measure $\mu_1\times\mu_2$ on $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$.

The argument is very similar to the proof of Theorem 8.6 in Rudin, a streamlined version of which was presented in class on 11/14. Let

$$\mathcal{F}=\{E\in\sigma(\mathcal{F}_1\times\mathcal{F}_2)\colon \mu(E)=\mu_1\times\mu_2(E)\}.$$

Then,

(1) $\mathcal{F}_1\times\mathcal{F}_2\subset\mathcal{F}$;

(2) $\mathcal{F}$ is closed under finite disjoint unions;

(3) if $E_1,E_2,\ldots\in\mathcal{F}$ and $E_1\subset E_2\subset\ldots$, then $\bigcup_{n=1}^{\infty}E_n\in\mathcal{F}$;

(4) if $X'\times Y'\in\mathcal{F}_1\times\mathcal{F}_2$, $E_1,E_2,\ldots\in\mathcal{F}$, $X'\times Y'\supset E_1\supset E_2\supset\ldots$, and $\mu_1(X'),\mu_2(Y')<\infty$, then $\bigcap_{n=1}^{\infty}E_n\in\mathcal{F}$.

The first property holds by (11); the other three hold by standard properties of measures.

Since $(X,\mathcal{F}_1,\mu_1)$ and $(Y,\mathcal{F}_2,\mu_2)$ are σ-finite, there exist $X_1,X_2,\ldots\in\mathcal{F}_1$ and $Y_1,Y_2,\ldots\in\mathcal{F}_2$ be such that

$$X_1\subset X_2\subset\ldots,\qquad X=\bigcup_{n=1}^{\infty}X_n,\qquad \mu_1(X_n)<\infty\ \forall\,n,$$
$$Y_1\subset Y_2\subset\ldots,\qquad Y=\bigcup_{n=1}^{\infty}Y_n,\qquad \mu_2(Y_n)<\infty\ \forall\,n.$$

For each $n\in\mathbb{Z}^+$, let

$$\mathcal{C}_n=\{E\in\sigma(\mathcal{F}_1\times\mathcal{F}_2)\colon E\cap(X_n\times Y_n)\in\mathcal{F}\}.$$

By (1) and (2) above, $\mathcal{C}_n$ contains the collection $\mathcal{R}$ consisting of finite unions of disjoint measurable rectangles (i.e. of elements of $\mathcal{F}_1\times\mathcal{F}_2$). By (3) and (4), $\mathcal{C}_n$ is a monotone class. Since $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$ is the minimal monotone class on $X\times Y$ containing $\mathcal{R}$, it follows that $\mathcal{C}_n=\sigma(\mathcal{F}_1\times\mathcal{F}_2)$ for every $n\in\mathbb{Z}^+$, i.e.

$$E_n\equiv E\cap(X_n\times Y_n)\in\mathcal{F}\qquad \forall\ E\in\sigma(\mathcal{F}_1\times\mathcal{F}_2),\ n\in\mathbb{Z}^+.$$

Since $E_1 \subset E_2 \subset \ldots$, (3) above then implies that

$$E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{F} \qquad \forall\, E \in \sigma(\mathcal{F}_1 \times \mathcal{F}_2),$$

i.e. $\mu(E) = \mu_1 \times \mu_2(E)$ for all $E \in \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$.

Problem 4 (8pts)

Let $\mathbb{I} = [0,1]$, $(\mathbb{I}, \mathcal{M}_{\mathbb{I}}, m_{\mathbb{I}})$ *be the usual Lebesgue measure space, and* $(\mathbb{I}, 2^{\mathbb{I}}, \mu)$ *be the measure space so that* μ *is the counting measure. Let*

$$E = \big\{(x,x) : x \in \mathbb{I}\big\} \subset \mathbb{I} \times \mathbb{I}$$

be the diagonal. Show that

$$E \in \sigma\big(\mathcal{M}_{\mathbb{I}} \times 2^{\mathbb{I}}\big), \qquad \varphi_E(x) \equiv \mu(E_x) = 1 \ \ \forall\, x \in \mathbb{I}, \qquad \psi_E(y) \equiv m_{\mathbb{I}}(E^y) = 0 \ \ \forall\, y \in \mathbb{I}.$$

Why doesn't this contradict equation (6.3) in the book?

Since

$$E = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{k} \left[\frac{n-1}{k}, \frac{k}{n}\right] \times \left[\frac{n-1}{k}, \frac{k}{n}\right]$$

is a countable intersection of finite unions of measurable rectangles (i.e. elements of $\mathcal{M}_{\mathbb{I}} \times 2^{\mathbb{I}}$), $E \in \sigma(\mathcal{M}_{\mathbb{I}} \times 2^{\mathbb{I}})$. Since $E_x = \{x\}$ and $E^y = \{y\}$,

$$\varphi_E(x) \equiv \mu(\{x\}) = 1, \quad \psi_E(y) \equiv m_{\mathbb{I}}(\{y\}) = 0, \qquad \int_X \varphi_E \, \mathrm{d}m_{\mathbb{I}} = 1, \quad \int_Y \psi_E \, \mathrm{d}\mu = 0.$$

This does not contradict equation (6.3) in the book because the measure space $(\mathbb{I}, 2^{\mathbb{I}}, \mu)$ is not σ-finite.

Problem 5 (12pts)

Let $\mathbb{I} = [0,1]$ *and* $(\mathbb{I}, \mathcal{M}_{\mathbb{I}}, m_{\mathbb{I}})$ *be the usual Lebesgue measure space. For each* $n \in \mathbb{Z}^+$, *let* $f_n : \mathbb{I} \longrightarrow \mathbb{R}$ *be a continuous function such that*

$$f_n(x) = 0 \ \ \forall\, x \notin \big[2^{-n}, 2^{-n+1}\big], \qquad \int_0^1 f_n(x)\, \mathrm{d}x = 1.$$

Show that

(a) *the sum* $f(x,y) = \sum_{n=1}^{\infty} \big(f_n(x) - f_{n+1}(x)\big) f_n(y)$ *converges for all* $(x,y) \in \mathbb{I}^2$ *and the function* $f : \mathbb{I}^2 \longrightarrow \mathbb{R}$ *is continuous except at* $(0,0)$;

(b) $\displaystyle\int_0^1\Big(\int_0^1 f(x,y)\,\mathrm{d}y\Big)\mathrm{d}x=1$ *and* $\displaystyle\int_0^1 f(x,y)\,\mathrm{d}x=0$ *for all* $y\in\mathbb{I}$. *Why doesn't this contradict any of the Fubini theorems?*

(a; **4pts**) If $y\in[2^{-k},2^{-k+1}]$, $f(x,y)=(f_k(x)-f_{k+1}(x))f_k(y)$, i.e. at most one term in the sum for $f(x,y)$ is nonzero and so the sum converges. Every $(x,y)\neq(0,0)$ has neighborhood $U_{x,y}\subset[0,1]^2$ so that at most 3 terms in the sum are nonzero on $U_{x,y}$. Since f is a sum of three continuous functions on $U_{x,y}$, it is continuous on $U_{x,y}$. Since these neighborhoods cover the complement of (0,0) in $[0,1]^2$, f is continuous on this complement.

(b; **8pts**) If $y\in[2^{-k},2^{-k+1}]$, $f(x,y)=(f_k(x)-f_{k+1}(x))f_k(y)$ and so

$$\int_0^1 f(x,y)\,\mathrm{d}x=\Big(\int_0^1 f_k(x)\,\mathrm{d}x-\int_0^1 f_{k+1}(x)\,\mathrm{d}x\Big)f_k(y)=(1-1)f_k(y)=0,$$
$$\int_0^1\Big(\int_0^1 f(x,y)\,\mathrm{d}x\Big)\mathrm{d}y=\int_0^1 0\,\mathrm{d}y=0.$$

If $x\in[2^{-1},1]$, $f(x,y)=f_1(x)f_1(y)$ and so

$$\int_0^1 f(x,y)\,\mathrm{d}y=f_1(x)\int_0^1 f_1(y)\,\mathrm{d}y=f_1(x).$$

If $x\in[2^{-k},2^{-k+1}]$ with $k>1$, $f(x,y)=f_k(x)(f_k(y)-f_{k-1}(y))$ and so

$$\int_0^1 f(x,y)\,\mathrm{d}x=f_k(x)\Big(\int_0^1 f_k(x)\,\mathrm{d}y-\int_0^1 f_{k-1}(x)\,\mathrm{d}y\Big)=f_k(x)(1-1)=0.$$

Combining the last two statements, we obtain

$$\int_0^1\Big(\int_0^1 f(x,y)\,\mathrm{d}y\Big)\mathrm{d}x=\int_{2^{-1}}^1 f_1(x)\,\mathrm{d}x=\int_0^1 f_1(x)\,\mathrm{d}x=1.$$

This does not contradict any of the Funini theorems because f changes sign and

$$\begin{aligned}\iint_{[0,1]^2}|f|\mathrm{d}\,m\times m&=\int_0^1\Big(\int_0^1\big|f(x,y)\big|\,\mathrm{d}x\Big)\mathrm{d}y\\
&\ge\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\Big(\int_{2^{-k}}^{2^{-k+1}}\big|f(x,y)\big|\,\mathrm{d}x+\int_{2^{-k-1}}^{2^{-k}}\big|f(x,y)\big|\,\mathrm{d}x\Big)\mathrm{d}y\\
&=\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\big|f_k(y)\big|\Big(\int_{2^{-k}}^{2^{-k+1}}\big|f_k(x)\big|\,\mathrm{d}x+\int_{2^{-k-1}}^{2^{-k}}\big|f_{k+1}(x)\big|\,\mathrm{d}x\Big)\mathrm{d}y\\
&=\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\big|f_k(y)\big|\Big(\int_0^1\big|f_k(x)\big|\,\mathrm{d}x+\int_0^1\big|f_{k+1}(x)\big|\,\mathrm{d}x\Big)\mathrm{d}y\\
&\ge\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}\big|f_k(y)\big|(1+1)\mathrm{d}y=2\sum_{k=1}^{\infty}\int_0^1|f_k(y)|\mathrm{d}y\ge2\sum_{k=1}^{\infty}1=\infty,\end{aligned}$$

i.e. f is not $m\times m$-integrable.