MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 1

Problem 1 (5pts)

Is the function

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2^n x)$$

Riemann integrable on $[0, 2\pi]$*? Justify your answer.*

Yes, because the finite sum

$$f_N(x) \equiv \sum_{n=0}^{n=N} 2^{-n} \sin(2^n x)$$

of continuous functions is Riemann integrable on $[0, 2\pi]$ for each $N \in \mathbb{Z}^+$ and

$$\big|f(x) - f_N(x)\big| \le \sum_{n=N+1}^{\infty} 2^{-n} \big|\sin(2^n x)\big| \le 2^{-N}. \tag{1}$$

Given $\varepsilon \in \mathbb{R}^+$, let $N \in \mathbb{Z}^+$ be such that $2\pi \cdot 2^{-N} < \frac{\varepsilon}{4}$ and let

$$0 = a_0 < a_1 < \ldots < a_m = 2\pi$$

be a partition of $[0, 2\pi]$ so that

$$\sum_{i=1}^{m} \big(\max_{[a_{i-1}, a_i]} f_N(x) - \min_{[a_{i-1}, a_i]} f_N(x)\big)\big(a_i - a_{i-1}\big) < \frac{\varepsilon}{2}.$$

Combining this with (1), we obtain

$$\begin{aligned} \sum_{i=1}^{m} \big(\max_{[a_{i-1}, a_i]} f(x) - \min_{[a_{i-1}, a_i]} f(x)\big)\big(a_i - a_{i-1}\big) &\le \sum_{i=1}^{m} \big(\max_{[a_{i-1}, a_i]} f_N(x) - \min_{[a_{i-1}, a_i]} f_N(x) + 2 \cdot 2^{-N}\big)\big(a_i - a_{i-1}\big) \\ &\le \frac{\varepsilon}{2} + 2 \cdot 2^{-N} \cdot 2\pi \le \varepsilon\,. \end{aligned}$$

This implies that $f(x)$ is Riemann integrable.

Remark: The part of the solution ending with (1) is enough for full credit. A less direct way to proceed from (1) is the following. By (1), f is a sum of continuous functions that converges uniformly. By Theorem 25.5 in Ross's textbook, f is thus continuous. By Theorem 33.2 in Ross's textbook, f is therefore integrable on $[0, 2\pi]$.

Problem 2 (10pts)

Show that the set

$$A=\left\{x\in[-1,1]\colon \left|\sin(nx)\right|\le\frac{1}{n}\ \forall\, n\in\mathbb{Z}^+\right\}$$

has measure 0.

For each $n\in\mathbb{Z}^+$, let

$$\begin{aligned}A_n&\equiv\left\{x\in[-1,1]\colon \left|\sin(nx)\right|\le\frac{1}{n}\right\}\\&=\left\{x\in[-1,1]\colon \pi k-\arcsin(1/n)\le nx\le \pi k+\arcsin(1/n)\text{ for some }k\in\mathbb{Z}\right\}\\&=\bigcup_{k=-n}^{k=n}\left\{x\in[-1,1]\colon \pi k/n-\arcsin(1/n)/n\le x\le \pi k/n+\arcsin(1/n)/n\right\}.\end{aligned}$$

Thus,

$$A=\bigcap_{n=1}^{\infty}A_n \qquad\text{and}\qquad m^*(A_n)\le 2(2n+1)\arcsin(1/n)/n\,. \tag{2}$$

Since $\sin(x)/x\longrightarrow 1$ as $x\longrightarrow 0$, $\arcsin(x)/x\longrightarrow 1$ as $x\longrightarrow 0$. Thus, there exists $N\in\mathbb{Z}^+$ so that $\arcsin(1/n)<2/n$ for all $n>N$ and

$$m^*(A_n)\le 2(2+1/n)\cdot 2/n\le 12/n \qquad \forall\ n>N. \tag{3}$$

Let $\varepsilon\in\mathbb{R}^+$ and $n\in\mathbb{Z}^+$ be such that $n>N$ and $12/n<\varepsilon$. By the first statement in (2) and by (3),

$$m^*(A)\le m^*(A_n)\le 12/n<\varepsilon.$$

This implies that A is a null set.

Problem 3 (10pts)

Let $A\subset[0,1]$ be a null subset. Show that

$$B\equiv\left\{x^2\colon x\in A\right\}$$

is also a null subset. Is the conclusion still true if $A\subset\mathbb{R}$? Justify your answer.

Let $\varepsilon\in\mathbb{R}^+$ and $I_1,I_2,\ldots$ be a sequence of intervals in $[0,1]$ such that

$$A\subset\bigcup_{k=1}^{\infty}I_k \qquad\text{and}\qquad \sum_{k=1}^{\infty}\ell(I_k)<\frac{\varepsilon}{2}. \tag{4}$$

If $a_k,b_k\in\mathbb{R}$ are the left and right endpoints of I_k, then

$$B\subset\bigcup_{k=1}^{\infty}\left\{x^2\colon x\in I_k\right\}=\bigcup_{k=1}^{\infty}\left[a_k^2,b_k^2\right], \qquad \sum_{k=1}^{\infty}\ell\big([a_k^2,b_k^2]\big)\le\sum_{k=1}^{\infty}2\ell(I_k)<\varepsilon. \tag{5}$$

The first equality above holds because x^2 is non-decreasing function on $[0,1]$. The second equality holds because

$$\ell\big([a_k^2,b_k^2]\big) = b_k^2 - a_k^2 = (b_k+a_k)(b_k-a_k) \le 2\ell\big([a_k,b_k]\big). \tag{6}$$

By (5), B is a null set.

The conclusion is true for every null subset $A\subset\mathbb{R}$. First suppose $A\subset[n-1,n]$ for some $n\in\mathbb{Z}^+$. With the notation as in the previous paragraph, (6) would remain valid if 2 on its right-hand side were replaced by $2n$. Thus, (5) would remain valid if $I_1,I_2,\ldots$ were chosen so that the second statement in (4) held with ε/n instead of ε and $2\ell(I_k)$ in (5) were replaced by $2n\ell(I_k)$. Thus, the conclusion holds for every null subset $A\subset[n-1,n]$ for some $n\in\mathbb{Z}^+$. It also holds for every null subset $A\subset[-n,-(n-1)]$ for some $n\in\mathbb{Z}^+$, because the subset

$$-A \equiv \big\{-x\colon x\in A\big\} \subset \big[n-1,n\big]$$

is then also null and has the same associated subset B.

For an arbitrary null subset $A\subset\mathbb{R}$ and $n\in\mathbb{Z}$, let

$$A_n = A\cap[n-1,n], \qquad B_n = \big\{x^2\colon x\in A_n\big\}.$$

Since A is a null subset, so is A_n. By the previous paragraph, B_n is then also a null subset. Since

$$B \equiv \big\{x^2\colon x\in A\big\} = \bigcup_{n=1}^{\infty}\big\{x^2\colon x\in A_n\big\} = \bigcup_{n=1}^{\infty}B_n$$

is a countable union of null subsets, it is also a null subset.

Problem 4 (10pts)

In Definition 2.3 in the textbook, the set Z_A involves sums taken over sequences of intervals I_n of every possible type (closed, open, open on lower/upper end and closed on upper/lower end). Show that using only one of these four kinds of intervals in the definition of Z_A would not change the definition of $m^*(A)$. This means that you need to establish 4 statements (e.g. using open interval in place of interval in the definition of Z_A would not change the answer, etc.)

Let $A\subset\mathbb{R}$. Denote by Z_A the set of numbers as in Definition 2.3 using all types of intervals, by Z_A° the set of numbers as in Definition 2.3 using only open intervals, and by Z_A' the set of numbers as in Definition 2.3 using only one of the four types of interval. Since $Z_A\supset Z_A'$ and $Z_A'\supset Z_A^\circ$ (b/c every open interval is contained in an interval of any given type of the same length),

$$m^*(A) \equiv \inf Z_A \le \inf Z_A' \le \inf Z_A^\circ\,.$$

It thus suffices to show that

$$\inf Z_A^\circ \le m^*(A)+\varepsilon \qquad \forall\,\varepsilon\in\mathbb{R}^+. \tag{7}$$

Let $\varepsilon \in \mathbb{R}^+$ and $I_1, I_2, \ldots$ be a sequence of intervals such that

$$A \subset \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} \ell(I_k) < m^*(A) + \frac{\varepsilon}{2}. \tag{8}$$

If $a_k, b_k \in \mathbb{R}$ are the left and right endpoints of I_k, let

$$J_k = \big(a_k - \varepsilon/2^{k+2}, b_k + \varepsilon/2^{k+2}\big).$$

By (8),

$$A \subset \bigcup_{k=1}^{\infty} J_k, \quad \sum_{k=1}^{\infty} \ell(J_k) < \sum_{k=1}^{\infty}\Big(\ell(I_k) + \frac{\varepsilon}{2^{k+1}}\Big) = \sum_{k=1}^{\infty} \ell(I_k) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} < m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m^*(A) + \varepsilon.$$

Thus, a number less than $m^*(A) + \varepsilon$ lies in Z_A°. This implies (7).

Problem 5 (15pts)

Let $F\colon [0,1] \longrightarrow [0,1]$ be the Lebesgue function defined at the top of p20 in the textbook. Show that $F(0)=0$, $F(1)=1$, and F is non-decreasing, continuous, and constant on each open interval removed in the construction of the Cantor set on p19 and takes a null set to a set of outer measure 1 (the book contains an outline for justifying these statements).

We use the same notation as in the book. For $x=0$, $a_n=0$ for all $n \in \mathbb{Z}^+$. For $x=1$, $a_n=2$ for all $n \in \mathbb{Z}^+$. Thus,

$$F(0) = \sum_{n=1}^{\infty} \frac{0/2}{2^n} = 0, \qquad F(1) = \sum_{n=1}^{\infty} \frac{2/2}{2^n} = 1.$$

Suppose $x < x'$ and $a_1, a_2, \ldots$ and $a_1', a_2', \ldots$ are the expansions of x and x'. Let $k \in \mathbb{Z}^+$ be the smallest index such $a_k < a_k'$ (thus $a_n = a_n'$ for all $n < k$). If $a_k = 1$, then

$$F(x) = \sum_{n=1}^{k-1} \frac{a_n/2}{2^n} + \frac{1}{2^k} = \sum_{n=1}^{k} \frac{a_n'/2}{2^n} \le F(x').$$

If $a_k = 0$, then

$$F(x) \le \sum_{n=1}^{k-1} \frac{a_n/2}{2^n} + \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{k-1} \frac{a_n'/2}{2^n} + \frac{1}{2^k} \le F(x').$$

Thus, $F(x) \le F(x')$ in either case and so F is non-decreasing.

Any interval removed at the k-th step of the construction of the Cantor set on p19 is of the form (a,b) with

$$a = .a_1 \ldots a_{k-1}1, \quad b = .a_1 \ldots a_{k-1}2 = .a_1 \ldots a_{k-1}1222\ldots \quad \text{for some } \; a_1, \ldots, a_{k-1} \in \{0,2\}.$$

The first index N for which $a_N\!=\!1$ is k for every $x\!\in\!(a,b)$. Thus,

$$F(x) = \sum_{n=1}^{k-1} \frac{a_n/2}{2^n} + \frac{1}{2^k} \qquad \forall\, x\!\in\!(a,b),$$

and so F is constant on each open interval removed in the construction of the Cantor set.

We now show that F is continuous at $x\!\in\![0,1]$. Let $\varepsilon\!\in\!\mathbb{R}^+$. Suppose first that $x\!\in\!(0,1)$ and thus $F(x)\!\in\!(0,1)$. Choose

$$k \in \mathbb{Z}^+, \quad a_1,\ldots,a_k,b_1,\ldots,b_k \in \{0,1\} \qquad \text{s.t.}$$
$$a_k,b_k=1, \quad A \equiv \sum_{n=1}^{k} \frac{a_n}{2^n} \in \big(F(x)\!-\!\varepsilon, F(x)\big), \quad B \equiv \sum_{n=1}^{k} \frac{b_n}{2^n} \in \big(F(x), F(x)\!+\!\varepsilon\big).$$

Let

$$a = \sum_{n=1}^{k-1} \frac{2a_n}{2^n} + \frac{1}{2^k}, \quad b = \sum_{n=1}^{k-1} \frac{2b_n}{2^n} + \frac{1}{2^k}\,.$$

Thus, $F(a)\!=\!A$, $F(b)\!=\!B$, and so

$$F(x)\!-\!\varepsilon < F(a) < F(x) < F(b) < F(x)\!+\!\varepsilon.$$

Since F is non-decreasing, this implies that

$$x \in (a,b) \qquad \text{and} \qquad (a,b) \subset F^{-1}\big((F(x)\!-\!\varepsilon, F(x)\!+\!\varepsilon)\big).$$

If $x\!=\!0$, we take $a\!=\!x$ above, so that

$$x \in [0,b) \qquad \text{and} \qquad [0,b) \subset F^{-1}\big([0,\varepsilon)\big).$$

If $x\!=\!1$, we take $b\!=\!x$ above, so that

$$x \in (a,1] \qquad \text{and} \qquad (a,1] \subset F^{-1}\big((1\!-\!\varepsilon,1)\big).$$

Thus, $F^{-1}(U)\!\subset\![0,1]$ is an open subset of $[0,1]$ for every open subset $U\!\subset\![0,1]$, i.e. F is continuous.

Since F is continuous and constant on each open interval removed in the construction of the Cantor set C, $F(C)\!=\![0,1]$. This provides an example of a null subset of $[0,1]$ taken by F to a set of outer measure 1.