MAT 324: Real Analysis, Fall 2017
Solutions to Midterm

Problem 1 (10pts)

(a) *Give a definition of σ-field on a set X.*

(b) *Describe all σ-fields on the set $X \equiv \{a,b\}$ of two elements; explain why there are no others.*

(a; **5pts**) A σ-field on X is a collection $\mathcal{F}$ of subsets of X such that

$$\text{(i) } X \in \mathcal{F} \quad \text{(ii) if } E \in \mathcal{F}, \text{ then } E^c \equiv X - E \in \mathcal{F} \quad \text{(iii) if } E_1, E_2, \ldots \in \mathcal{F}, \text{ then } \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$$

(b; **5pts**) The σ-fields on the set $X \equiv \{a,b\}$ are

$$\mathcal{F} = \{\emptyset, X\} \qquad \text{and} \qquad \mathcal{F} = 2^X = \{\emptyset, X, \{a\}, \{b\}\}.$$

By (i) and (ii) above, every σ-field $\mathcal{F}$ on X contains X and $X^c = \emptyset$. If $\mathcal{F}$ also contains $\{a\}$, then it contains $\{a\}^c = \{b\}$ as well. So, there are no other σ-fields on X.

Problem 2 (10pts)

Let $(X, \mathcal{F}, \mu)$ be a measure space.

(a) *Given a definition of what it means for a function $f\colon X \longrightarrow \mathbb{R}$ to be measurable.*

(b) *Suppose $f, g\colon X \longrightarrow \mathbb{R}$ are measurable functions and $A, B \in \mathcal{F}$ are disjoint subsets such that $A \cup B = X$. Show that the function*

$$h\colon X \longrightarrow \mathbb{R}, \qquad h(x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \end{cases}$$

is measurable.

(a; **3pts**) A function $f\colon X \longrightarrow \mathbb{R}$ is measurable if $f^{-1}(I) \in \mathcal{F}$ for every interval $I \subset \mathbb{R}$.

(b; **7pts**) Let $I \subset \mathbb{R}$ be an interval. Then,

$$h^{-1}(I) = \big(f^{-1}(I) \cap A\big) \cup \big(g^{-1}(I) \cap B\big) \subset X.$$

Since f and g are measurable functions, $f^{-1}(I), g^{-1}(I) \in \mathcal{F}$. Since $\mathcal{F}$ is a σ-field, it is closed under countable intersections and unions. Since $A, B \in \mathcal{F}$, it follows that

$$f^{-1}(I) \cap A,\, g^{-1}(I) \cap B \in \mathcal{F} \qquad \Longrightarrow \qquad h^{-1}(I) \in \mathcal{F}.$$

Problem 3 (20pts)

Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a smooth function, $A\subset\mathbb{R}$ be a null subset, and

$$f(A)\equiv\big\{f(x)\colon x\in A\big\}.$$

(a) *Show that $f(A)$ is a null set if A is bounded (i.e. $A\subset[-R,R]$ for some $R\in\mathbb{R}^+$).*

(b) *Show that $f(A)$ is a null set (whether or not A is bounded).*

(a; **15pts**) Suppose $A\subset[-R,R]$. Let

$$M=\max_{[-R,R]}\big\{|f'(x)|\colon x\in[a,b]\big\}+1.$$

By the Fundamental Theorem of Calculus or the Mean Value Theorem,

$$\big|f(x)-f'(x)\big|\le M|x-x'| \qquad \forall\,x,x'\in[-R,R]. \tag{1}$$

Let $\varepsilon>0$. Since A is a null set, there exists a countable collection $\{I_n\}_{n\in\mathbb{Z}^+}$ of intervals such that

$$A\subset\bigcup_{n=1}^{\infty}I_n, \qquad \sum_{n=1}^{\infty}\ell(I_n)<\frac{\varepsilon}{M}\,.$$

Along with (1), this implies that

$$f(A)\subset f\bigg(\bigcup_{n=1}^{\infty}I_n\bigg)=\bigcup_{n=1}^{\infty}f(I_n)=\bigcup_{n=1}^{\infty}\big[\min_{I_n}f,\max_{I_n}f\big],$$
$$\sum_{n=1}^{\infty}\ell\big(\big[\min_{I_n}f,\max_{I_n}f\big]\big)\le\sum_{n=1}^{\infty}M\ell(I_n)<M\cdot\frac{\varepsilon}{M}=\varepsilon.$$

Thus, A is a null set.

(b; **5pts**) Let $A_n=A\cap[-n,n]$. By (a), $f(A_n)$ is a null set. Thus,

$$f(A)=f\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)=\bigcup_{n=1}^{\infty}f(A_n)$$

is also a null set.

Note: It is not sufficient to assume that f is continuous. The Lebesgue function F defined on page 20 is continuous, non-decreasing, and takes the Cantor set C (which is a null set) onto the entire interval $[0,1]$; see Problem 5 on PS1.

Problem 4 (20pts)

Denote by $\mathcal{M}\subset 2^{\mathbb{R}}$ *the collection of Lebesgue measurable subsets and by* $\mathcal{B}\subset\mathcal{M}$ *the collection of Borel subsets. Let* $A\subset\mathbb{R}$*. Show that*

(a) $A\in\mathcal{M}$ *if and only if* $\inf\big\{m^*(B-A)\colon B\in\mathcal{B},\, A\subset B\big\}=0$;

(b) $A\in\mathcal{M}$ *if and only if* $\inf\big\{m^*(A-B)\colon B\in\mathcal{B},\, B\subset A\big\}=0$.

If $A\in\mathcal{M}$, there exist $C,\mathcal{O}\subset\mathbb{R}$ such that

$$C,\mathcal{O}\in\mathcal{B},\quad C\subset A\subset\mathcal{O},\quad m^*(A-C),m^*(\mathcal{O}-A)=0.$$

This is basically Theorems 2.17 and 2.29; C is a countable union of closed sets, while $\mathcal{O}$ is a countable intersection of open sets. Thus,

$$\begin{aligned}\inf\big\{m^*(B-A)\colon B\in\mathcal{B},\, A\subset B\big\} &= m^*(\mathcal{O}-A)=0,\\ \inf\big\{m^*(A-B)\colon B\in\mathcal{B},\, B\subset A\big\} &= m^*(A-C)=0.\end{aligned}$$

This establishes one direction in each part.

Suppose $\inf\big\{m^*(B-A)\colon B\in\mathcal{B},\, A\subset B\big\}=0$. For each $n\in\mathbb{Z}^+$, let $B_n\in\mathcal{B}$ be such that $A\subset B_n$ and $m^*(B_n-A)<1/n$. Thus,

$$B\equiv\bigcap_{n=1}^{\infty}B_n\in\mathcal{B},\quad A\subset B,\quad m^*(B-A)\le m^*(B_n-A)\le\frac{1}{n}\quad\forall\, n\in\mathbb{Z}^+.$$

Thus, $B-A$ is a null set and so $B-A\in\mathcal{M}$. Since $\mathcal{M}$ is closed under differences,

$$A=B-(B-A)\in\mathcal{M}\,.$$

This establishes the other direction in (a).

Suppose $\inf\big\{m^*(A-B)\colon B\in\mathcal{B},\, B\subset A\big\}=0$. For each $n\in\mathbb{Z}^+$, let $B_n\in\mathcal{B}$ be such that $B_n\subset A$ and $m^*(A-B_n)<1/n$. Thus,

$$B\equiv\bigcup_{n=1}^{\infty}B_n\in\mathcal{B},\quad B\subset A,\quad m^*(A-B)\le m^*(A-B_n)\le\frac{1}{n}\quad\forall\, n\in\mathbb{Z}^+.$$

Thus, $A-B$ is a null set and so $A-B\in\mathcal{M}$. Since $\mathcal{M}$ is closed under countable unions,

$$A=B\cup(A-B)\in\mathcal{M}\,.$$

This establishes the other direction in (b).

Problem 5 (20pts)

For each $n\in\mathbb{Z}^+$, *define*

$$f_n, g_n\colon [0,\infty)\longrightarrow\mathbb{R}, \qquad f_n(x)=\frac{n^2x\mathrm{e}^{-nx}}{1+x^2}, \quad g_n(x)=\frac{x\mathrm{e}^{-x}}{1+x^2/n^2}\,.$$

(a) *Find* $\displaystyle\int_0^\infty\big(\lim_{n\longrightarrow\infty}f_n\big)\mathrm{d}x$ *and* $\displaystyle\int_0^\infty\big(\lim_{n\longrightarrow\infty}g_n\big)\mathrm{d}x$.

(b) *Show that*

$$\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty f_n\mathrm{d}x\bigg)=\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty g_n\mathrm{d}x\bigg)$$

and find this limit.

(c) *Let* $F\colon[0,\infty)\longrightarrow[0,\infty]$ *be a Lebesgue measurable function such that* $f_n\le F$ *for all* $n\in\mathbb{Z}^+$. *Show that*

$$\int_{[0,1]}F\mathrm{d}m=\infty\,.$$

(a; **5pts**) Since $f_n(0)=0$ for all n, $f_n(0)\longrightarrow0$. Since e^{nx} with $x>0$ dominates every polynomial in n as $n\longrightarrow\infty$, $f_n(x)\longrightarrow0$ for all $x>0$ as well. It is immediate that

$$g_n(x)\longrightarrow\frac{x\mathrm{e}^{-x}}{1+0}=x\mathrm{e}^{-x}\qquad\text{as } n\longrightarrow\infty.$$

Thus,

$$\int_0^\infty\big(\lim_{n\longrightarrow\infty}f_n\big)\mathrm{d}x=\int_0^\infty0\mathrm{d}x=0,\quad \int_0^\infty\big(\lim_{n\longrightarrow\infty}g_n\big)\mathrm{d}x=\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x=\big(-x\mathrm{e}^{-x}-\mathrm{e}^{-x}\big)\Big|_0^\infty=1.$$

(b; **8pts**) By the change of variables $x\longrightarrow nx$,

$$\int_0^\infty f_n\mathrm{d}x=\int_0^\infty\frac{(nx)\mathrm{e}^{-(nx)}}{1+(nx)^2/n^2}\mathrm{d}(nx)=\int_{n\cdot0}^{n\cdot\infty}\frac{x\mathrm{e}^{-x}}{1+x^2/n^2}\mathrm{d}x=\int_0^\infty g_n\mathrm{d}x\,.$$

This implies that the two limits in the statement are the same. Since $g_n(x)\ge0$ and $g_n(x)\nearrow x\mathrm{e}^{-x}$ for all $x\in[0,\infty)$,

$$\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty g_n\mathrm{d}x\bigg)=\lim_{n\longrightarrow\infty}\bigg(\int_{[0,\infty)}g_n\mathrm{d}m\bigg)=\int_{[0,\infty)}\big(\lim_{n\longrightarrow\infty}g_n\big)\mathrm{d}m=\int_{[0,\infty)}x\mathrm{e}^{-x}\mathrm{d}m=\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x=1;$$

the second equality above holds by the Monotone Convergence Theorem.

(c; **7pts**) Since $f_n\ge0$, the assumption implies that $|f_n|\le F$ for all $n\in\mathbb{Z}^+$. If in addition $\displaystyle\int_{[0,1]}F\mathrm{d}m<\infty$, then

$$\lim_{n\longrightarrow\infty}\bigg(\int_0^\infty f_n\mathrm{d}x\bigg)=\int_0^\infty\big(\lim_{n\longrightarrow\infty}f_n\big)\mathrm{d}x=0.$$

by the Dominated Convergence Theorem and part (a). However, this contradicts part (b).

Problem 6 (20pts)

(a) *State a definition of what it means for a bounded function $f\colon [0,1] \longrightarrow [0,\infty)$ to be* Riemann integrable.

(b) *State a definition of what it means for a bounded function $f\colon [0,1] \longrightarrow [0,\infty)$ to be* Lebesgue integrable.

(c) *Give an example of a bounded Lebesgue integrable function $f\colon [0,1] \longrightarrow [0,\infty)$ which is not Riemann integrable. Justify your answer.*

(d) *Show that every bounded Riemann integrable function $f\colon [0,1] \longrightarrow [0,\infty)$ is also Lebesgue integrable.*

(a; **5pts**) For a partition $(0=a_0<a_1<\ldots<a_k=1)$ of $[0,1]$, let

$$s_P(f) = \sum_{n=1}^{k}\Big(\inf_{[a_{n-1},a_n]} f\Big)\big(a_n-a_{n-1}\big), \qquad S_P(f) = \sum_{n=1}^{k}\Big(\sup_{[a_{n-1},a_n]} f\Big)\big(a_n-a_{n-1}\big).$$

The bounded function $f\colon [0,1]\longrightarrow[0,\infty)$ is Riemann integrable if

$$\sup\big\{s_P(f)\colon P \text{ is partition of } [0,1]\big\} = \inf\big\{S_P(f)\colon P \text{ is partition of } [0,1]\big\}.$$

(b; **5pts**) A bounded function $f\colon [0,1]\longrightarrow[0,\infty)$ is Lebesgue integrable if

$$\int_{[0,1]} f\mathrm{d}m \equiv \sup\bigg\{\sum_{n=1}^{k} a_n m(A_n)\colon\ A_n\in\mathcal{M},\ A_n\subset[0,1],\ A_n\cap A_{n'}=\emptyset\ \forall\, n\neq n',\ a_n\in\mathbb{R}^{\ge 0},\ a_n\le\inf_{A_n} f\bigg\}$$
$$<\infty.$$

(c; **5pts**) By Theorem 4.33(i), a bounded Riemann integrable function $f\colon [0,1]\longrightarrow[0,\infty)$ is a.e. continuous. Since the bounded function

$$f=1_{[0,1]\cap\mathbb{Q}}\colon \mathbb{Q}\longrightarrow[0,\infty), \qquad f(x)=\begin{cases}1, & \text{if } x\in[0,1]\cap\mathbb{Q};\\ 0, & \text{if } x\in[0,1]-\mathbb{Q};\end{cases}$$

is nowhere continuous, it is not Riemann integrable. Since $f=0$ a.e. on $[0,1]$ and the constant function 0 is Lebesgue integrable, so is the function f.

(d; **5pts**) By Theorem 4.33(i), a bounded Riemann integrable function $f\colon [0,1]\longrightarrow[0,\infty)$ is a.e. continuous and is thus measurable. On the other hand,

$$0\le \int_{[0,1]} f\mathrm{d}m \le \big(\sup_{[0,1]} f\big)m\big([0,1]\big) = \sup_{[0,1]} f < \infty,$$

since f is bounded. Thus, f is Lebesgue integrable on $[0,1]$.