MAT 324: Real Analysis, Fall 2017
Midterm (80 mins)

Please write your solutions on the provided printer paper, starting each problem on a new page and clearly indicating the number (and part) of the problem you are solving. You do *not* need to copy the statements of the problems. You can cite established statements from the textbook as appropriate (e.g. not statements you are being asked to establish). There are 6 problems on this exam of varying difficulty.

Problem 1 (10pts)

(a) Give a definition of σ-field on a set X.

(b) Describe all σ-fields on the set $X \equiv \{a, b\}$ of two elements; explain why there are no others.

Problem 2 (10pts)

Let $(X, \mathcal{F}, \mu)$ be a measure space.

(a) Given a definition of what it means for a function $f\colon X \longrightarrow \mathbb{R}$ to be measurable.

(b) Suppose $f, g\colon X \longrightarrow \mathbb{R}$ are measurable functions and $A, B \in \mathcal{F}$ are disjoint subsets such that $A \cup B = X$. Show that the function

$$h\colon X \longrightarrow \mathbb{R}, \qquad h(x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \end{cases}$$

is measurable.

Note: You can take $(X, \mathcal{F}, \mu)$ to be the standard Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$. This makes no difference, and there will be no grading penalty.

Problem 3 (20pts)

Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function, $A \subset \mathbb{R}$ be a null subset, and

$$f(A) \equiv \big\{ f(x)\colon x \in A \big\}.$$

(a) Show that $f(A)$ is a null set if A is bounded (i.e. $A \subset [-R, R]$ for some $R \in \mathbb{R}^+$).

(b) Show that $f(A)$ is a null set (whether or not A is bounded).

Note: If you cannot do (a) as stated, you can try it with f non-decreasing or $f(x) = x^3$ at the cost of 2 or 5 points penalty, respectively, to begin with. For the purposes of (b), you can assume (a).

Problem 4 (20pts)

Denote by $\mathcal{M} \subset 2^{\mathbb{R}}$ the collection of Lebesgue measurable subsets and by $\mathcal{B} \subset \mathcal{M}$ the collection of Borel subsets. Let $A \subset \mathbb{R}$. Show that

(a) $A \in \mathcal{M}$ if and only if $\inf\big\{m^*(B-A)\colon B \in \mathcal{B},\ A \subset B\big\} = 0$;

(b) $A \in \mathcal{M}$ if and only if $\inf\big\{m^*(A-B)\colon B \in \mathcal{B},\ B \subset A\big\} = 0$.

Problem 5 (20pts)

For each $n \in \mathbb{Z}^+$, define

$$f_n, g_n \colon [0,\infty) \longrightarrow \mathbb{R}, \qquad f_n(x) = \frac{n^2 x \mathrm{e}^{-nx}}{1+x^2}, \quad g_n(x) = \frac{x\mathrm{e}^{-x}}{1+x^2/n^2}.$$

(a) Find $\displaystyle\int_0^\infty \big(\lim_{n\longrightarrow\infty} f_n\big)\mathrm{d}x$ and $\displaystyle\int_0^\infty \big(\lim_{n\longrightarrow\infty} g_n\big)\mathrm{d}x$.

(b) Show that

$$\lim_{n\longrightarrow\infty}\left(\int_0^\infty f_n \mathrm{d}x\right) = \lim_{n\longrightarrow\infty}\left(\int_0^\infty g_n \mathrm{d}x\right)$$

and find this limit.

(c) Let $F\colon [0,\infty) \longrightarrow [0,\infty]$ be a Lebesgue measurable function such that $f_n \le F$ for all $n \in \mathbb{Z}^+$. Show that

$$\int_{[0,1]} F \mathrm{d}m = \infty\,.$$

Problem 6 (20pts)

(a) State a definition of what it means for a bounded function $f\colon [0,1] \longrightarrow [0,\infty)$ to be Riemann integrable.

(b) State a definition of what it means for a bounded function $f\colon [0,1] \longrightarrow [0,\infty)$ to be Lebesgue integrable.

(c) Give an example of a bounded Lebesgue integrable function $f\colon [0,1] \longrightarrow [0,\infty)$ which is not Riemann integrable. Justify your answer.

(d) Show that every bounded Riemann integrable function $f\colon [0,1] \longrightarrow [0,\infty)$ is also Lebesgue integrable.