MAT 324: Real Analysis, Fall 2017
Homework Assignment 8

Please read carefully Sections 8.1, 5.2, 5.4, and 5.5 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 8 (**due in class on Thursday, 11/9**): Problems 1-3 below and on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner-product space over $\mathbb{C}$ and $e_1, e_2, \ldots \in V$ be a sequence of orthonormal vectors, i.e.

$$\langle e_i, e_j\rangle = \begin{cases} 1, & \text{if } i=j; \\ 0, & \text{if } i\neq j. \end{cases} \tag{1}$$

Let $|\cdot|$ be the norm on V determined by $\langle\cdot,\cdot\rangle$. Show that

(a) the collection $\{e_i : i \in \mathbb{Z}^+\}$ is linearly independent (no non-trivial *finite* linear combination of e_i's with *complex* coefficients adds up to 0);

(b) $v - \sum_{i=1}^{i=k} \langle v, e_i\rangle e_i$ is orthogonal to $e_1, \ldots, e_k$ for all $v \in V$;

(c) $\sum_{i=1}^{i=k} \big|\langle v, e_i\rangle\big|^2 \le |v|^2$ for all $v \in V$ and $k \in \mathbb{Z}$;

(d) the sequence $v_k \equiv \sum_{i=1}^{i=k} \langle v, e_i\rangle e_i$ is Cauchy in $(V, \langle\cdot,\cdot\rangle)$.

Problem 2

Let $\mathbb{I}=[0,1]$, $L^2(\mathbb{I};\mathbb{C})$ be the L^2-space of $\mathbb{C}$-valued functions on $\mathbb{I}$, and $f\in L^2(\mathbb{I};\mathbb{C})$. For each $n\in\mathbb{Z}$, define

$$\psi_n\colon \mathbb{I}\longrightarrow\mathbb{C},\quad \psi_n(x)=\mathrm{e}^{2\pi\mathrm{i}\,nx}, \qquad c_n(f)=\langle\!\langle f,\psi_n\rangle\!\rangle_2\equiv\int_{\mathbb{I}} f\overline{\psi}_n\mathrm{d}m\in\mathbb{C}. \tag{2}$$

(a) Show that the collection $\{\psi_n\colon n\in\mathbb{Z}^+\}$ consists of orthonormal elements of $L^2(\mathbb{I};\mathbb{C})$.

(b) Show that the sum

$$\sum_{n\in\mathbb{Z}} c_n(f)\psi_n \equiv \lim_{k,m\longrightarrow\infty}\sum_{n=-k}^{n=m} c_n(f)\psi_n$$

converges with respect to the L^2-norm to some $h_f\in L^2(\mathbb{I};\mathbb{C})$ so that $\|h_f\|_2\le\|f\|_2$ and $\langle\!\langle f-h_f,\psi_n\rangle\!\rangle_2=0$ for all $n\in\mathbb{Z}$.

(c) Suppose f is twice continuously differentiable, $f(0)=f(1)$, and $f'(0)=f'(1)$. Show that

$$c_n(f)=\frac{1}{2\pi\mathrm{i}\,n}c_n(f')=-\frac{1}{4\pi^2 n^2}c_n(f'') \qquad \forall\, n\neq 0.$$

(d) Under the assumptions on f in (c), show that the sum in (b) converges uniformly to a continuous function $h_f\colon\mathbb{I}\longrightarrow\mathbb{C}$.

Hint: for (c), use integration by parts

Problem 3

Let $\mathbb{I}=[0,1]$, $L^2(\mathbb{I};\mathbb{C})$ be the L^2-space of $\mathbb{C}$-valued functions on $\mathbb{I}$, $f\in L^2(\mathbb{I};\mathbb{C})$, and ψ_n and $c_n(f)$ be as in (2).

(a) Suppose f is continuous and $f(0)=f(1)$. Show that for every $\epsilon>0$ there exist $N\in\mathbb{Z}^+$ and $a_n\in\mathbb{C}$ with $n\in\mathbb{Z}$ such that

$$\bigg|f(x)-\sum_{n=-N}^{n=N}a_n\psi_n(x)\bigg|\le\epsilon \qquad \forall\, x\in\mathbb{I}.$$

(b) Suppose f is twice continuously differentiable, $f(0)=f(1)$, and $f'(0)=f'(1)$. Show that

$$f=\sum_{n\in\mathbb{Z}}c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nx} \tag{3}$$

with the sum converging uniformly on $\mathbb{I}$.

Hint: deduce (a) from the Weierstrass approximation theorem and (b) from (a) applied to $f-h_f$ with h_f as in (d) of the previous problem.

Note: This says that the Fourier series of a function f as in (b) converges to f uniformly on $\mathbb{I}$. If f is smooth and all its derivatives at 0 and 1 agree, then a similar argument shows that

$$\begin{aligned}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}f&=\sum_{n\in\mathbb{Z}}\big(2\pi\mathrm{i}n\big)^\ell c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nx}\equiv\lim_{k,m\longrightarrow\infty}\sum_{n=-k}^{n=m}\big(2\pi\mathrm{i}n\big)^\ell c_n(f)\psi_n\\&=\lim_{k,m\longrightarrow\infty}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\sum_{n=-k}^{n=m}c_n(f)\psi_n\end{aligned}$$

i.e. the Fourier series of f converges to f uniformly with all derivatives.