MAT 324: Real Analysis, Fall 2017
Homework Assignment 5

Please read carefully Sections 4.2 and 4.3 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 5 (**due in class on Thursday, 10/12**): Problems 1-5 on below

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let $(X,\mathcal{F},\mu)$ be a measure space and $f_n : X \longrightarrow [0,\infty]$ be a sequence of measurable functions. Show that

$$\int_X\bigg(\sum_{n=1}^{\infty} f_n\bigg)\mathrm{d}\mu = \sum_{n=1}^{\infty}\bigg(\int_X f_n \mathrm{d}\mu\bigg).$$

Problem 2

Let $(X,\mathcal{F},\mu)$ be a measure space and $f_n : X \longrightarrow [0,\infty]$ be a sequence of measurable functions decreasing almost everywhere to $f: X \longrightarrow [0,\infty]$. Suppose $\displaystyle\int_X f_1 \mathrm{d}\mu < \infty$. Show that

$$\int_X f \mathrm{d}\mu = \lim_{n\longrightarrow\infty}\int_X f_n \mathrm{d}\mu\,.$$

Hint: proof of Theorem 2.19(ii) might be helpful

Problem 3

(a) Let $(X,\mathcal{F},\mu)$ be a measure space and $f: X \longrightarrow [0,\infty)$ be a measurable function. For $n\in\mathbb{Z}$, define

$$E_n = \big\{x\in X : 2^n < f(x) \le 2^{n+1}\big\}.$$

Show that f is integrable on X if and only if $\displaystyle\sum_{n\in\mathbb{Z}} 2^n\mu(E_n) < \infty$.

(b) Let $a\in\mathbb{R}$. Use (a) to show that the function $f(x)=x^{-a}$ is Lebesgue integrable on $(0,1)$ if and only if $a<1$.

Problem 4

Let $(X,\mathcal{F},\mu)$ be a measure space and $f: X \longrightarrow \overline{\mathbb{R}}$ be a measurable function which is integrable on X. Show that $f\ge 0$ almost everywhere on X if and only if $\int_E f\mathrm{d}\mu \ge 0$ for all $E\in\mathcal{F}$.
Hint: proof of Theorem 4.22 might be helpful

Problem 5

Show that the limit $\displaystyle\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} x^n \mathrm{e}^{-n|x|}\mathrm{d}m$ exists and find it.