MAT 324: Real Analysis, Fall 2017
Homework Assignment 4

Please read carefully Sections 3.4-3.6,4.1 and page 303 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 4 (**due in class on Thursday, 10/05**): Problems 1-5 on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

(a) Write each $x \in [0,1]$ as $x = \sum_{n=1}^{\infty} \frac{a_n(x)}{2^n}$ with each $a_n(x) \in \{0,1\}$ taking the infinite expansion for all $x \neq 0$. Show that the function $a_n \colon [0,1] \longrightarrow \mathbb{R}$ is measurable.

(b) Show that the function

$$f\colon [0,1] \longrightarrow \mathbb{R}, \qquad f(x) = \sum_{n=1}^{\infty} \frac{2a_n(x)}{3^n},$$

is measurable, injective, and takes values in the Cantor set C.

Problem 2

(a) Let $f : X \longrightarrow Y$ be any map, $\mathcal{S} \subset 2^Y$, and $\mathcal{F}_Y \subset 2^Y$ be the σ-field generated by $\mathcal{S}$ (i.e. the smallest σ-field on Y containing $\mathcal{S}$). Show that

$$\mathcal{F}_X \equiv \{ f^{-1}(B) \colon B \in \mathcal{F}_Y \}$$

is the σ-field generated by $\{f^{-1}(B) \colon B \in \mathcal{S}\}$.

(b) Let $(X, \mathcal{F}, \mu)$ be a measure space and $f : X \longrightarrow \mathbb{R}$ be a measurable function. Show that $f^{-1}(B) \in \mathcal{F}$ for every Borel subset $B \subset \mathbb{R}$ (i.e. $B \in \mathcal{B}$).

(c) Give an example of a measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a measurable subset $E \subset \mathbb{R}$ (i.e. $E \in \mathcal{M}$) so that $f^{-1}(E)$ is not measurable.

(d) Give an example of measurable functions $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ so that $f \circ g \colon \mathbb{R} \longrightarrow \mathbb{R}$ is not measurable.

Hint: p303 might be helpful

Problem 3

Let $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Show that the set

$$E'_f \equiv \{ x \in \mathbb{R} \colon f \text{ is differentiable at } x \}$$

is measurable.

Problem 4

A car leaves point A at random between 1pm and 2pm and travels at 50mph towards point B, which is 20 miles away. Find the probability distribution of the distance at 1:50pm.

Problem 5

Let $F \colon [0,1] \longrightarrow [0,1]$ be the Lebesgue function defined at the top of p20. Find $\int_{[0,1]} F \mathrm{d}m$.