MAT 324: Real Analysis, Fall 2017
Homework Assignment 11

Please read carefully Sections 7.1 and 7.2 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below.

Problem Set 11 (**due in class on Thursday, 12/7**): Problems 1-4 below on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let μ, ν_1, ν_2 be measures on a measurable space $(X, \mathcal{F})$. Show that

(a) if $\nu_1, \nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.

(b) if $\nu_1, \nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.

(c) if $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.

Problem 2

Let μ, ν be measures on a measurable space $(X, \mathcal{F})$.

(a) Suppose for every $\epsilon > 0$ there exists $\delta > 0$ such that $\nu(E) < \epsilon$ for every $E \in \mathcal{F}$ with $\mu(E) < \delta$. Show that $\nu \ll \mu$.

(b) Show that the converse is true if $\nu(X) < \infty$.

(c) Give an example showing that the converse can fail if $\mu(X) < \infty$ and $\nu(X) = \infty$.

Problem 3

Let m be the standard Lebesgue measure on $(\mathbb{R}, \mathcal{M})$ and μ be the counting measure on $(\mathbb{R}, \mathcal{M})$. Show that $m \ll \mu$, but there exists no Lebesgue measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}^{\geq 0}$ such that $m = \mu_g$. Why doesn't this contradict the Radon-Nikodym theorem?

Problem 4

Let $\mathcal{F} = 2^{\mathbb{Z}^+}$ be the σ-field of all subsets of $\mathbb{Z}^+$ and

$$\mu, \nu \colon \mathcal{F} \longrightarrow \mathbb{R}, \qquad \mu(E) = \sum_{n \in E} 2^{-n}, \quad \nu(E) = \sum_{n \in E} 3^{-n}.$$

(a) Show that $\nu \ll \mu$ and $\mu \ll \nu$.

(b) Find the Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$.

(c) For $k, a \in \mathbb{Z}^+$, let

$$E_{k;a} = \{ n \in \mathbb{Z}^+ : n \equiv a \mod k \}.$$

Show that for each $k \in \mathbb{Z}^+$, the collection

$$\mathcal{P}_k \equiv \{ E_{k;a} : a = 1, 2, \ldots, k \}$$

is a finite partition of $\mathbb{Z}^+$ into measurable subsets.

(d) For each $k \in \mathbb{Z}^+$, find the associated function $h_{\mathcal{P}_k} : \mathbb{Z}^+ \longrightarrow \mathbb{R}$ as on page 191. Find the limit of this sequence of functions.