MAT 320: Introduction to Analysis
Homework Assignment 6

Please read Section 13 of Ross's textbook thoroughly before starting on the problem set below.
Optional supplementary reading: pp30-42 of Rudin's book

Problem Set 6 (**due at the start of recitation on Wednesday, 3/20**): D-G (below), 13.12, 13.14, 13.3ba, 13.3c (below), 13.15

Problem D

Let $d_{\mathbb{R}}$ be the standard metric on $\mathbb{R}$, $d_{\mathbb{R}}(x,x')=|x-x'|$. What are the closures in $(\mathbb{R},d_{\mathbb{R}})$ of

$$\mathbb{N}, \qquad S=\{2^m : m\in\mathbb{Z}\}, \qquad \text{and} \quad [-2,2]\cap\mathbb{Q} \quad ?$$

Note that $\infty\notin\mathbb{R}$. Give examples (3 in total) of a closed non-compact subset of $(\mathbb{R},d_{\mathbb{R}})$, of a bounded non-compact subset of $(\mathbb{R},d_{\mathbb{R}})$, and of a closed bounded non-compact subset of $(\mathbb{Q},d_{\mathbb{R}}|_{\mathbb{Q}})$. *Answers only.*

Problem E

Let (X,d) be a metric space, $\mathcal{C}$ be some collection of subsets of X (i.e. each *element* $B\in\mathcal{C}$ is a *subset* $B\subset X$), and $A=\bigcup_{B\in\mathcal{C}} B$.

(a) Show that $\overline{A}\supset\bigcup_{B\in\mathcal{C}}\overline{B}$, where $\overline{A},\overline{B}\subset X$ are the closures of A and B, respectively, in (X,d).

(b) Show that the opposite inclusion holds if $\mathcal{C}$ is finite, but may not hold if $\mathcal{C}$ is countable.

Problem F

Let d and d' be two metrics on the same set X that are uniformly equivalent: there exists $C\in\mathbb{R}^+$ such that

$$C^{-1}d(x,x')\le d'(x,x')\le Cd(x,x') \qquad \forall\, x,x'\in X.$$

(a) Show that a subset $U\subset X$ is open/closed w.r.t. d if and only if $U\subset X$ is open/closed w.r.t. d'.

(b) Show that a sequence $(x_n)_n$ converges to x (resp. is Cauchy) w.r.t. d if and only if $(x_n)_n$ converges to x (resp. is Cauchy) w.r.t. d'.

(c) Show that (X,d) is bounded/complete/compact if and only if (X,d') is.

Problem G

Let (X,d_X) and (Y,d_Y) be two metric spaces. Define

$$d_1,d_2,d_3\colon (X\times Y)^2\longrightarrow\mathbb{R}, \quad d_i\big((x,y),(x',y')\big)=\begin{cases}\max(d_X(x,x'),d_Y(y,y')), & \text{if } i=1;\\ \big(d_X(x,x')^2+d_Y(y,y')^2\big)^{1/2}, & \text{if } i=2;\\ d_X(x,x')+d_Y(y,y'), & \text{if } i=3.\end{cases}$$

(a) Show that these three functions are metrics and that any two of them are uniformly equivalent.

(b) Take $(X,d_X),(Y,d_Y)=(\mathbb{R},d_{\mathbb{R}})$, i.e. $\mathbb{R}$ with the standard metric. On a *whole* page by itself, draw 3 *huge* (but separate) copies of the first quadrant of the xy-plane. On the i-th copy, clearly draw the open unit ball $B_1^i((2,2))$ around $(2,2)\in\mathbb{R}^2$ with respect to the metric d_i (make sure it comes out *large*). On the same copy, *clearly* indicate what it means for this ball to be also open with respect to the metric d_{i+1} (with $d_4\equiv d_1$), as F-(a) says should be the case. You can add a few words clarifying the diagrams, but they should be mostly clear by themselves.

Problem 13.3c

Show that the metric space (B,d) in 13.3a is complete.