MAT 319/320: Basics of Analysis, Spring 2018
Power Series (Review from Calculus C)

Extremely Important: power series

Very Important: radius and interval of convergence for power series; differentiation, integration, and limits of functions via power series; Taylor series

Important: finding radius and interval of convergence of power series; determining Taylor series of functions related to standard ones; applications of power series to computing sums of series

C: Power Series

C.1 A power series is a function defined by an infinite series of the form

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n\,, \tag{C1}$$

where a is some fixed number, typically 0, called the center of the power series (C1); plugging in $x=a$ makes all the terms $(x-a)^n$ to be 0^n. So the center of the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}$$

is $x=-1$. By convention used in defining power series, $(x-a)^0=1$ even if $x=a$. By a similar convention, $0!=1$ so that

$$\boxed{(n+1)!=n!\cdot(n+1)}$$

for all non-negative integers n.

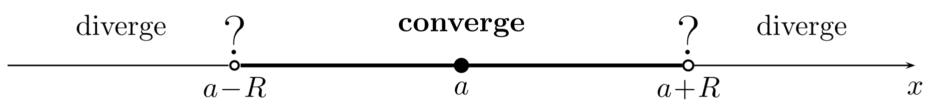
C.2 For each x for which the series (C1) converges, we obtain a number $f(x)$. In particular,

$$f(a)=c_0 0^0+c_1 0^1+c_2 0^2+\ldots=c_0;$$

so the power series (C1) **always** converges at its center $x=a$. The most fundamental question about a power series is the set of the numbers x for which the power series converges. By the Main Theorem about Power Series, this set can be of one of 3 types, with one type having 4 sub-types:

> The power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ converges either
>
> (a) for $x=a$ only;
>
> (b) for all x;
>
> (c) for x in one of the intervals $(a-R,a+R)$, $[a-R,a+R)$, $(a-R,a+R]$, or $[a-R,a+R]$ for some $R>0$ and diverges otherwise
>
> (C2)

The four possibilities in (c) are illustrated below:



According to this theorem, the set of values of x for which a power series converges cannot be arbitrary. It must be an interval, which is centered at the center of the power series, may consist of a single point, be infinite, or be of finite nonzero length and in the last case can be open, closed, or half-open (so the series can either converge or diverge at each of the two end-points of the interval; this is indicated by the question marks in the sketch). The interval on which a power series converges is its interval of convergence. The number R in (c) is the radius of convergence of the power series; $R=0$ in (a) and $R=\infty$ in (b).

C.3 In order to **find the radius of convergence** of a power series as in (C1) with $c_n \neq 0$ for all n ($\geq$ some N), use the *Ratio Test* with $a_n = c_n(x-a)^n \neq 0$ (so we are assuming $x \neq a$, since we already know that the power series converges for $x=a$):

$$\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\longrightarrow\infty} \frac{|c_{n+1}|\cdot|x-a|^{n+1}}{|c_n|\cdot|x-a|^n} = |x-a|\cdot \lim_{n\longrightarrow\infty} \frac{|c_{n+1}|}{|c_n|}\,. \tag{C3}$$

In general, the last limit in (C3) does not have to exist. However, in all examples encountered in this class it either exists or $|c_{n+1}|/|c_n| \longrightarrow \infty$. In these cases:

- if $|c_{n+1}|/|c_n| \longrightarrow \infty$, the last number in (C3) is ∞; by the general *Ratio Test* the series (C1) diverges for every $x \neq a$, and so we are in case (a) of (C2) and $R=0$.
- if $|c_{n+1}|/|c_n| \longrightarrow 0$, the last number in (C3) is 0; by the general *Ratio Test* the series (C1) converges for every $x \neq a$, and so we are in case (b) of (C2) and $R=\infty$.
- if $|c_{n+1}|/|c_n| \longrightarrow C \neq 0$, the last number in (C3) is $C|x–a|$; by the general *Ratio Test* the series (C1) converges if $|x-a| < 1/C$ and diverges $|x-a| > 1/C$. So we are in case (c) of (C2) and $R=1/C$.

Once the radius of convergence is found, **find the interval of convergence** of the power series. If $R=0$, then the interval of convergence is just $[a,a]$; if $R=\infty$, then the interval of convergence is $(-\infty,\infty)$. If $R \neq 0,\infty$, it remains to determine whether the power series converges for $x=a-R$ and for $x=a+R$, i.e. you have to determine *separately* whether each of the two power series

$$f(a-R) = \sum_{n=0}^{\infty} c_n(-R)^n = \sum_{n=0}^{\infty} c_n(-1)^n R^n \qquad \text{and} \qquad f(a+R) = \sum_{n=0}^{\infty} c_n R^n$$

converges. You will have to use some convergence/divergence tests for series, but **not** the *Ratio Test* (it would give 1 in the limit and so be inconclusive in these two cases). Once this is done, the interval of convergence will be as in one of the 4 subcases in (c) of (C2).

Remark 1: If c_n involves $n!$ in the *numerator* and the remaining terms are powers and exponentials of n, such as $\sqrt{n+1}$ or 2^n (but not n^n), then you'll be in case (a) of (C2). If c_n involves $n!$ in the *denominator* and the remaining terms are powers and exponentials of n, such as n^3 or 5^n (but not n^n), then you'll be in case (b) of (C2). If c_n involves *only* powers and exponentials of n, such as $n^3/\sqrt{n^2+n}$ or 3^n (but not n^n), then you'll be in case (c) of (C2). This is the case when you'll also need to determine whether the series converges or diverges at *each* of the two end-points of the interval of convergence *separately.*

Remark 2: By the above, if $|c_{n+1}|/|c_n| \longrightarrow C$ for some nonnegative number C or for $C=\infty$, then the radius of convergence of the power series is $R=1/C$. In general, $|c_{n+1}|/|c_n|$ may not approach anything, including ∞, because it may keep on jumping. For example, $|c_{n+1}|/|c_n|$ with n odd might approach 0 and $|c_{n+1}|/|c_n|$ with n even might approach ∞; then 0 and ∞ are said to be *limits of subsequences*. There can be lots of such limits of subsequences, but there is always at least one (possibly $\pm\infty$). The largest of such limits is denoted lim sup (and the smallest lim inf). If a sequence converges, lim sup is just its usual limit. If C is lim sup of the sequence $|c_{n+1}|/|c_n|$, then the radius of convergence of the power series (C1) is still $R=1/C$. You can learn more about lim sup in MAT 320.

C.4 If the radius of convergence of a power series is 0, the power series is rather useless. However, if its radius of convergence R is positive (possibly ∞), it defines a *smooth* function $f(x)$ on $(a-R, a+R)$. This function can be differentiated and integrated by differentiating and integrating the power series term by term (like a polynomial):

> If the radius of convergence R of the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ is positive (possibly ∞),
>
> the function $\displaystyle f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is smooth on the open interval $(a-R, a+R)$,
>
> - $\displaystyle f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$, the radius of convergence of this power series is still R, and if $R\neq\infty$ and the series for f *diverges* for $x=a\pm R$, so does the series for f';
> - $\displaystyle\int f(x)\mathrm{d}x = C + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1} = C + \sum_{n=1}^{\infty}\frac{c_{n-1}}{n-1}(x-a)^n$, the radius of convergence of this power series is still R, and if $R\neq\infty$ and the series for f *converges* for $x=a\pm R$, so does the series for $\displaystyle\int f(x)\mathrm{d}x$.
>
> (C4)

So the radius of convergence of a power series does not change under differentiation and integration, but the interval of convergence may change if $R\neq\infty$: differentiation may *remove* one or both of the endpoints from the interval of convergence, while integration may *add* them to the interval convergence. For example, the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \tag{C5}$$

converges if and only if $|x|<1$ and for each x such that $|x|<1$ it converges to $1/(1-x)$. So the radius of convergence of the series (C5) is 1, its interval of convergence is $(-1,1)$, and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \qquad \text{if} \quad |x|<1. \tag{C6}$$

Differentiating both sides of (C6) with respect to x gives

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \ldots \qquad \text{if} \quad |x|<1. \tag{C7}$$

The radius of convergence of this power series is still 1, while the interval of convergence is still $(-1,1)$ because there are no ends to potentially drop off from the interval of convergence of the power series (C6). Integrating both sides of (C6) from $x=0$ (this makes $C=0$ in (C4)) gives

$$-\ln(1-x) = \int_0^x \frac{1}{(1-u)}du = \sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{x^n}{n} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots. \tag{C8}$$

The radius of convergence of this power series is still 1, but the interval of convergence may have increased by either or both of the two endpoints; this has to be checked separately. Setting $x=1$ in the series in (C8) gives $\sum_{n=1}^{\infty}\frac{1}{n}$; this series diverges by the *p-Series Test.* Setting $x=-1$ in the series in (C8) gives $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$; this series converges by the *Alternating Series Test.* Thus, the interval of convergence of the power series in (C8) is $[-1,1)$ and

$$-\ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \qquad \text{if} \quad -1 \le x < 1. \tag{C9}$$

Taking $x=1/2$ and $x=-1$ in (C9) gives

$$\begin{aligned} -\ln(1/2) &= \sum_{n=1}^{\infty}\frac{(1/2)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n\,2^n} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \ldots, \\ -\ln(2) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \ldots, \end{aligned}$$

Since $-\ln(1/2)=\ln(2)$, we find that

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n\,2^n} = \ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}} \tag{C10}$$

Remark 1: You do not need to memorize the two formulas in (C10), but you need to understand and be able to apply the principles involved in obtaining them; in particular, you have to be able to find sums of analogous infinite series. You have to remember the formulas for differentiating and integrating power series given in (C4); remembering the first of the two formulas for each should suffice and should be fairly easy, since these are just differentiation and integration of (infinite) polynomials. If you are asked to find the sum of an infinite series as in (C10), you need to be able to see that it is obtained from some *power series* by replacing x with a specific value in the range of the convergence of x. You should then be able to recognize the power series and know what function it sums up to, at least after dropping same factors of n from all terms. By (C4), extra factors of n correspond to differentiation or integration of the power series you recognize (but be careful to check that the exponents of x are correct and not shifted by a fixed number; if they are shifted, just take a power of x outside of the summation). You can then determine the function to which the original power series corresponds and sum up the starting infinite series by evaluating this function at the appropriate value of x.

Remark 2: the statements of (C4) regarding the radii of convergence follow from Remark 2 in **C.3** above; so you can actually verify them assuming $|c_{n+1}|/|c_n| \longrightarrow C$ for some $C \ge 0$ (possibly ∞).

C.5 Limits of functions defined via power series can be easily computed, as long as the limit is taken at the center of the power series. This generally involves writing out the first few terms of the power series. For example, the function

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

is defined whenever $-1\le x\le 1$ and thus for all x close to 0; so it makes sense to ask about limits of related functions at $x\!=\!0$. In particular,

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{f(x)-x-\frac{1}{2}x^2}{x^3}=\lim_{x\longrightarrow 0}\frac{\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots\right)-x-\frac{1}{2}x^2}{x^3}&=\lim_{x\longrightarrow 0}\frac{\left(\frac{x^3}{3}+\frac{x^4}{4}+\ldots\right)}{x^3}\\&=\lim_{x\longrightarrow 0}\left(\frac{1}{3}+\frac{1}{4}x+\ldots\right)=\frac{1}{3}\,;\end{aligned}$$

on the second line $\ldots$ denotes terms involving x and higher powers of x, all of which approach 0 as $x\longrightarrow 0$. You can compute this limit using l'Hospital's rule as well, but it would have to be applied *3 times*, re-checking the required assumptions each time (in this case, this would mean checking that the numerator and denominator both approach 0).

C.6 Two power series with the same center, say 0,

$$f(x)=\sum_{n=0}^{\infty}f_nx^n \qquad\text{and}\qquad g(x)=\sum_{n=0}^{\infty}g_nx^n$$

can be multiplied together by treating them as infinite polynomials and collecting coefficients of each power of $(x\!-\!a)$:

$$\begin{aligned}f(x)g(x)&=\left(\sum_{n=0}^{\infty}f_nx^n\right)\left(\sum_{n=0}^{\infty}g_nx^n\right)\\&=\left(f_0+f_1x+f_2x^2+f_3x^3+\ldots\right)\left(g_0+g_1x+g_2x^2+g_3x^3+\ldots\right)\\&=f_0g_0+(f_0g_1+f_1g_0)x+(f_0g_2+f_1g_1+f_2g_0)x^2+(f_0g_3+f_1g_2+f_2g_1+f_3g_0)x^3+\ldots\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{k=n}f_kg_{n-k}\right)x^n.\end{aligned}$$

So the coefficient of x^n in the product is the sum of $n\!+\!1$ terms, each of which is a product of a term from the f-series and a term from the g-series. If the f and g-series converge for $|x|<R$, then so does the fg-series. For example,

$$\begin{aligned}\frac{1}{(1-x)^2}=\frac{1}{1-x}\cdot\frac{1}{1-x}&=\left(\sum_{n=0}^{\infty}x^n\right)\left(\sum_{n=0}^{\infty}x^n\right)\\&=1\cdot 1+(1\cdot 1+1\cdot 1)x+(1\cdot 1+1\cdot 1+1\cdot 1)x^2+\ldots=\sum_{n=1}^{\infty}nx^{n-1}\,;\end{aligned}$$

this agrees with (C7), as well as with the $k=-2$ case of (D7) below. A more interesting example is

$$\begin{aligned}\frac{1}{x^2-3x+2} &= \frac{1}{(x-1)(x-2)} = \frac{1}{1-x}\cdot\frac{1/2}{1-x/2}\\ &= \frac{1}{2}\Big(\sum_{n=0}^{\infty} x^n\Big)\Big(\sum_{n=0}^{\infty}\Big(\frac{x}{2}\Big)^n\Big) = \frac{1}{2}\Big(\sum_{n=0}^{\infty} x^n\Big)\Big(\sum_{n=0}^{\infty}\frac{x^n}{2^n}\Big)\\ &= \frac{1}{2}\bigg(1\cdot 1+\Big(1\cdot\frac{1}{2}+1\cdot 1\Big)x+\Big(1\cdot\frac{1}{4}+1\cdot\frac{1}{2}+1\cdot 1\Big)x^2+\ldots\bigg)\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1^{n+1}-(1/2)^{n+1}}{1-1/2}x^n = \sum_{n=0}^{\infty}\Big(1-\frac{1}{2^{n+1}}\Big)x^n\end{aligned}$$

The second-to-last equality above is the $(a,b)=(1,1/2)$ case of

$$\begin{aligned}a^{n+1}-b^{n+1} &= (a-b)\big(a^n b^0+a^{n-1}b^1+\ldots+a^1b^{n-1}+a^0b^n\big)\\ &= (a-b)\big(b^n+ab^{n-1}+\ldots+a^{n-1}b+a^n\big);\end{aligned}$$

this formula was used to sum up geometric series. Another way to expand $1/(x^2-3x+2)$ around $x=0$ is to use *partial fractions* and addition of series, instead of multiplication (addition is much simpler):

$$\begin{aligned}\frac{1}{x^2-3x+2} &= \frac{1}{(x-1)(x-2)} = \frac{1}{(-2)-(-1)}\Big(\frac{1}{x-1}-\frac{1}{x-2}\Big) = -\frac{1}{x-1}+\frac{1}{x-2}\\ &= \frac{1}{1-x}-\frac{1/2}{1-x/2} = \sum_{n=0}^{\infty}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{x}{2}\Big)^n = \sum_{n=0}^{\infty}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\Big(1-\frac{1}{2^{n+1}}\Big)x^n\end{aligned}$$

The interval of convergence of this power series can easily be seen to be $(-1,1)$.

D: Taylor Series

D.1 There is *at most* one way to expand a function into a power series centered at a given point:

> If $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ on $(a-R,a+R)$ for some $c_0,c_1,\ldots$ and $R>0$, then
>
> - f is a smooth function on $(a-R,a+R)$;
> - $c_n=\dfrac{f^{\langle n\rangle}(a)}{n!}$, where $f^{\langle n\rangle}=f^{\langle n\rangle}(x)$ is the n-th derivative of f, $f^{\langle 0\rangle}=f$, $0!=1$.

(D1)

A power series expansion of f around $x=a$, if it exists, is called the Taylor series expansion of f around $x=a$. By the first statement in (D1), the function $f(x)=|x|$ does not admit a Taylor series expansion around $x=a$, though it does admit such an expansion around any $x=a\neq 0$:

$$|x|=\begin{cases}a+1\cdot(x-a) \text{ on } (0,\infty), & \text{if } a>0;\\ -a-1\cdot(x-a) \text{ on } (-\infty,0), & \text{if } a<0.\end{cases}$$

D.2 The second statement in (D1) provides a method of determining the Taylor coefficients c_n of f at $x=a$. However, this method is practical only if *all* derivatives of f can be computed. This can be done in some cases, including

(TS0) $f(x) = p(x)$ is a polynomial of degree d; then

$$p(x) = p(a) + p'(a)(x-a) + \frac{p''(a)}{2!}(x-a)^2 + \ldots + \frac{p^{\langle d\rangle}(a)}{d!}(x-a)^d\,, \qquad \text{(D2)}$$

because $p^{\langle n\rangle}(x)=0$ if $n>d$. The "series" on the right-hand side of (D2) converges for all x because it is a finite sum. The equality of the left and right expressions in (D2) also holds for all x, because of *Taylor's Inequality* below (not because the right expression is a finite sum).

(TS1) $f(x) = 1/(1-x)$: in this case $f^{\langle n\rangle}(x)=n!/(1-x)^{n+1}$ as can be seen by induction, and so

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}\frac{n!/1^{n+1}}{n!}x^n = \sum_{n=0}^{\infty}x^n \qquad \text{if} \quad |x|<1. \qquad \text{(D3)}$$

The series on the right-hand side is a geometric series with $r=x$. It converges if and only if $|x|<1$; if so, it converges to $1/(1-x)$ as stated in (D3).

(TS2) $f(x)=\mathrm{e}^x$: in this case $f^{\langle n\rangle}(x)=\mathrm{e}^x$ for all n and

$$\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}\frac{\mathrm{e}^0}{n!}x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,. \qquad \text{(D4)}$$

The series on the right-hand side converges for all x by the Ratio Test. The equality of the left and right expressions in (D4) also holds for all x, because of *Taylor's Inequality* below (not because the series converges for all x).

(TS3) $f(x)=\cos x$: in this case

$$f^{\langle 4n\rangle}(x) = \cos x\,, \quad f^{\langle 4n+1\rangle}(x) = -\sin x\,, \quad f^{\langle 4n+2\rangle}(x) = -\cos x\,, \quad f^{\langle 4n+3\rangle}(x) = \sin x$$

for all n, as can be seen by induction. Since $\cos 0 = 1$ and $\sin 0 = 0$,

$$\cos x = \sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}\frac{(-1)^n\cos 0}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\,. \qquad \text{(D5)}$$

The series on the right-hand side converges for all x by the Ratio Test. The equality of the left and right expressions in (D5) also holds for all x, because of *Taylor's Inequality* below (not because the series converges for all x).

(TS4) $f(x)=\sin x$: in this case

$$f^{\langle 4n\rangle}(x) = \sin x\,, \quad f^{\langle 4n+1\rangle}(x) = \cos x\,, \quad f^{\langle 4n+2\rangle}(x) = -\sin x\,, \quad f^{\langle 4n+3\rangle}(x) = -\cos x$$

for all n, as can be seen by induction. Since $\cos 0 = 1$ and $\sin 0 = 0$,

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}\frac{(-1)^n\cos 0}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(D6)}$$

The series on the right-hand side converges for all x by the Ratio Test. The equality of the left and right expressions in (D6) also holds for all x, because of *Taylor's Inequality* below (not because the series converges for all x).

(TS5) $f(x) = (1+x)^k$, k is any real number and $|x| < 1$ (so that $(1+x)^k$ is defined even if k is not integer): in this case

$$f^{\langle n\rangle}(x) = k(k-1)(k-2)\dots(k-n+1)(1+x)^{k-n}$$

for all n, as can be seen by induction. Thus,

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \binom{k}{n}(1+0)^{k-n}x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad \text{if} \quad |x| < 1, \quad \text{(D7)}$$

$$\text{where} \qquad \binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}.$$

The radius of convergence of the series on the right-hand side is 1 by the Ratio Test, unless k is a non-negative integer (in which case RHS of (D7) is a finite sum and so converges for all x). The equality of the left and right expressions in (D7) also holds for $|x|<1$, because of 8.7 69 (not because the series converges if $|x| < 1$). The identity (D7) is known as binomial series. The geometric series (D3) is a special case of (D7), with $k=-1$ and x replaced by $-x$. If k is a non-negative integer, so that $(1+x)^k$ is a polynomial, for all $n>k$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots0\dots(k-n+1)}{n!} = 0$$

and so the series (D7) has only finitely many terms, and the identity holds for all x.

D.3 In many cases, it is not practical to compute all derivatives of a function and so it may not be possible to use the formula in (D1) to compute the Taylor coefficients directly. However, it may be possible to obtain the Taylor expansion for a given function by using one of the "standard" series (D3)-(D6). For example,

$$x^5 e^{-3x^2} = x^5 \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = x^5 \sum_{n=0}^{\infty} \frac{(-3)^n (x^2)^n}{n!} = x^5 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+5}}{n!}.$$

Since the Taylor series for e^x converges for all x, so does the above Taylor series for $x^5 e^{-3x^2}$. Similarly,

$$\frac{x^5}{1+3x^2} = \frac{x^5}{1-(-3x^2)} = x^5 \sum_{n=0}^{\infty} (-3x^2)^n = x^5 \sum_{n=0}^{\infty} (-3)^n x^{2n} = \sum_{n=0}^{\infty} (-3)^n x^{2n+5}.$$

Since the Taylor series for $1/(1-x)$ converges if $|x|<1$, the above Taylor series converges if $|-3x^2| < 1$ (whatever is used for x in the power series also has to be used in the bound for convergence); so it converges if $|x|<1/\sqrt{3}$.

Remark 1: When you use a Taylor series for one function to get a Taylor series expansion for another function, make sure your final answer is a power series in x (or $(x-a)$ if the center $a \neq 0$), not a power series in, say, $-3x^2$ or x^2, and not a product of a power series with, say, x^5 (see the two examples above). While there are many different ways to describe a function, there is at most **one way** to write it as a power series.

Remark 2: You should remember the formula for the Taylor coefficients c_n in (D1) or equivalently the general Taylor expansion formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n.$$

this formula is often used with $a=0$. On the other hand, the four formulas (D3)-(D6) will be provided for the exam (see the last page of *Final Exam Information*). You can take these four power series expansions, along with their intervals of convergence, as given and use them as appropriate. They may be helpful in obtaining other Taylor series, computing limits, and computing sums of infinite series. For full credit, you must derive any other power series formula you use on the exam, either directly from the Taylor coefficient formula in (D1) or from one of the four given Taylor series. In particular, you may need to derive a formula like (D2) for a specific polynomial $p(x)$ around specified center $x=a$; see 8.7 11,12 for examples. You should *not* memorize the binomial formula (D7); if anything related, with a specific k, appears on the exam, you should not quote the binomial formula anyway.

D.4 The key to determining whether a given function admits Taylor series expansion around a given point is Taylor's Inequality:

$$\left|f(x)-\sum_{n=0}^{n=m}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n\right|\le C_{m+1}(f;R)\frac{R^{m+1}}{(m+1)!}\qquad \text{if } \ a-R\le x\le a+R, \tag{D8}$$

where $C_{m+1}(f;R)$ is the maximum value of $|f^{\langle m+1\rangle}(x)|$ with x in $[a-R,a+R]$. This inequality is obtained as follows. Let

$$R_{m+1}(x)=f(x)-\sum_{n=0}^{n=m}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n.$$

Then,

$$R_{m+1}(a)=0,\quad R'_{m+1}(a)=0,\quad \ldots,\quad R^{\langle m\rangle}_{m+1}(a)=0.$$

Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
R^{\langle m\rangle}_{m+1}(x)&=R^{\langle m\rangle}_{m+1}(a)+\int_a^x R^{\langle m+1\rangle}_{m+1}(u)\mathrm{d}u=\int_a^x R^{\langle m+1\rangle}_{m+1}(u)\mathrm{d}u\\
R^{\langle m-1\rangle}_{m+1}(x)&=R^{\langle m-1\rangle}_{m+1}(a)+\int_a^x R^{\langle m\rangle}_{m+1}(u)\mathrm{d}u=\int_a^x R^{\langle m\rangle}_{m+1}(u)\mathrm{d}u\\
&\vdots\\
R_{m+1}(x)=R^{\langle 0\rangle}_{m+1}(x)&=R^{\langle 0\rangle}_{m+1}(a)+\int_a^x R^{\langle 1\rangle}_{m+1}(u)\mathrm{d}u=\int_a^x R^{\langle 1\rangle}_{m+1}(u)\mathrm{d}u
\end{aligned}$$

Thus, for all x in $[a,a+R]$:

$$\begin{aligned}
\big|R^{\langle m\rangle}_{m+1}(x)\big|&\le\int_a^x|R^{\langle m+1\rangle}_{m+1}(u)|\mathrm{d}u\le\int_a^x C_{m+1}(f;R)\mathrm{d}u\le C_{m+1}(f;R)|x-a|\\
\big|R^{\langle m-1\rangle}_{m+1}(x)\big|&\le\int_a^x|R^{\langle m\rangle}_{m+1}(u)|\mathrm{d}u\le\int_a^x C_{m+1}(f;R)|x-a|\mathrm{d}u\le\frac{C_{m+1}(f;R)}{2!}|x-a|^2\\
&\vdots\\
\big|R^{\langle 0\rangle}_{m+1}(x)\big|&\le\int_a^x|R^{\langle 1\rangle}_{m+1}(u)|\mathrm{d}u\le\int_a^x\frac{C_{m+1}(f;R)}{m!}|x-a|^m\mathrm{d}u\le\frac{C_{m+1}(f;R)}{(m+1)!}|x-a|^{m+1}
\end{aligned}$$

The same estimates holds if x lies in $[a-R,a]$. This confirms (D8). By (D8), if

$$\lim_{m\longrightarrow\infty}C_n(f;R)\frac{R^n}{n!}=0,$$

then

$$f(x)=\lim_{m\longrightarrow\infty}\sum_{n=0}^{n=m}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n\,,$$

and so the Taylor series for f does indeed equal to f on $[a-R,a+R]$. In particular, this is the case if

- $f(x)=p(x)$ is a polynomial of degree d: since $p^{\langle n\rangle}(x)=0$ if $n>d$, $C_n(f;R)=0$;
- $f(x)=\mathrm{e}^x$, $a=0$: since $f^{\langle n\rangle}(x)=\mathrm{e}^x$ for all n, $C_n(f;R)=\mathrm{e}^R$;
- $f(x)=\cos x$ or $f(x)=\sin x$: since $f^{\langle n\rangle}(x)$ is $\pm\cos x$ or $\pm\sin x$, $C_n(f;R)=1$ if $R\ge\pi$.

D.5 Power/Taylor series can be used to compute sums of some convergent infinite series, $\sum\limits_{n=0}^{\infty}a_n$, and even check convergence (in some cases only). Begin by writing the infinite series as evaluation of some power series at some point:

$$\sum_{n=0}^{\infty}a_n=\sum_{n=0}^{\infty}c_nx^n\Big|_{x=b}\,.$$

So you need to guess an appropriate sequence $\{c_n\}$ and the evaluation point b, but in some cases they may be evident. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n\,2^n}=\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{1}{n}x^n\Big|_{x=\frac{1}{2}}\,,\qquad \sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}nx^n\Big|_{x=\frac{1}{2}}.$$

You next need to find a simple formula for the function

$$g(x)=\sum_{n=0}^{\infty}c_nx^n\,.$$

It may not be one of the standard Taylor series, but may become such after dropping a fraction involving powers of n. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n,\sum_{n=1}^{\infty}nx^n\quad\longrightarrow\quad\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\,.$$

In light of (C4), the function $g=g(x)$ can then be reconstructed from the function $f=f(x)$ through differentiation and/or integration and possibly multiplication by a power of x after each step to account for differences in the exponent if any. In the case of integration, the constant C has to be chosen appropriately as well. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n+C=\int\left(\sum_{n=1}^{\infty}x^{n-1}\right)\mathrm{d}x=\int\frac{\mathrm{d}x}{1-x}=-\ln(1-x)+C'\quad\Longrightarrow\quad\sum_{n=1}^{\infty}\frac{1}{n}x^n=-\ln(1-x);$$

$$\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\left(\sum_{n=0}^{\infty}x^n\right)'=x\left(\frac{1}{1-x}\right)'=\frac{x}{(1-x)^2};$$

the last equality on the first line is obtained by setting $x=0$. The interval of convergence for the g-series can be determined from the f-series. For example, the radii of convergence of both series

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n=-\ln(1-x)\,,\qquad\sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2},\tag{D9}$$

are 1, since this is the radius of convergence of the geometric series $\sum_{n=1}^{\infty} x^n$ and the *radius* of convergence of a power series does not change under differentiation or integration. Since the interval of convergence of $\sum_{n=1}^{\infty} x^n$ is $(-1,1)$, this is also the interval of convergence of the power series $\sum_{n=1}^{\infty} nx^n$ since differentiation can only remove (but not necessarily) the end-points from the interval of convergence (and there are not any end-points to remove in this case). On the other hand, integration can only add in the end-points. Since $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ converges for $x=-1$ and diverges at $x=1$, the interval of convergence of this power series is $[-1,1)$. Once it is established that the required evaluation point b lies inside of the interval of convergence of the g-series, we obtain

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n x^n \Big|_{x=b} = g(b).$$

For example, since $1/2$ lies in the intervals of convergence of the two power series in (D9),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n\,2^n} &= \sum_{n=1}^{\infty} \frac{1}{n} x^n \Big|_{x=\frac{1}{2}} = -\ln(1-x)\big|_{x=\frac{1}{2}} = -\ln(1/2) = \ln(2), \\ \sum_{n=1}^{\infty} \frac{n}{2^n} &= \sum_{n=1}^{\infty} nx^n \Big|_{x=\frac{1}{2}} = \frac{x}{(1-x)^2}\Big|_{x=\frac{1}{2}} = \frac{1/2}{1/4} = 2; \end{aligned}$$

make sure to simplify the final answer as much as possible.

Remark: If the g-series is already a standard series, you do not need to do any differentiation or integration: just plug in the evaluation point to get the sum, as long as the evaluation point lies in the interval of convergence of the power series. In all cases, if you actually know that the infinite series converges, the evaluation point automatically lies in the interval of convergence. If you are asked to justify that the series converges, you either need to use one of the convergence/divergence tests for series or show that the evaluation point lies inside of the interval of convergence. In order to do so, it is often sufficient to determine just the *radius* of convergence: if the distance from the center to the evaluation point is (strictly) less than the radius of convergence, than the evaluation point lies in the interval of convergence. This is the case in the two examples above, since the distance from 0 to $1/2$ is less than 1.

Warning: Be careful about the lower summation bound. For example,

$$\sum_{n=2}^{\infty} \frac{1}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n\,2^n} - \frac{1}{1\cdot 2^1} = -\ln(1-x)\big|_{x=\frac{1}{2}} - \frac{1}{2} = -\ln(1/2) - \frac{1}{2} = \ln(2) - \frac{1}{2}\,.$$