MAT 320 4/30/19

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Last topic: Riemann (/Darboux) Integral #1

Dfn 1: A partition $\mathcal{P}$ of $[a,b]$ is $\{a=x_0<x_1<\dots<x_n=b\}$

$f:[a,b]\to\mathbb{R}$ bounded function, $\sup|f|<\infty$; $\boxed{a<b}$

$\mathcal{P}=\{a=x_0<x_1\dots<x_n=b\}$ = partition of $[a,b]$

$\to L_{f;\mathcal{P}}\equiv\sum_{k=1}^{n}\left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k-x_{k-1})$ lower sum

$U_{f;\mathcal{P}}=\sum_{k=1}^{\infty}\left(\sup_{x\in[x_{k-1},x_k]} f(x)\right)(x_k-x_{k-1})$ upper sum

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Example 2: $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$

$\mathcal{P}=\{0=x_0<x_1<\dots\le x_n=1\}$

$L_{f;\mathcal{P}}=\sum_{k=1}^{n}\left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k-x_{k-1})=\sum_{k=1}^{n}0\cdot(x_k-x_{k-1})=0$

$[x_{k-1},x_k]-\mathbb{Q}\neq\emptyset$

$U_{f;\mathcal{P}}=\sum_{k=1}^{\infty}\left(\sup_{x\in[x_{k-1},x_k]} f(x)\right)(x_k-x_{k-1})=\sum_{k=1}^{n}1\cdot(x_k-x_{k-1})$

$=(x_1-x_0)+(x_2-x_1)+\dots+(x_n-x_{n-1})$

$=x_n-x_0=1$

Example 1: $f:[0,1]\to\mathbb{R}$, $f(x)=x$ #2

$\mathcal{P}_n=\{a=0<\frac{1}{n}<\dots<\frac{n}{n}=1\}\rightsquigarrow L_{f;\mathcal{P}_n}=\frac{1}{2}(1-\frac{1}{n})$, $U_{f;\mathcal{P}_n}=\frac{1}{2}(1+\frac{1}{n})$

$L_f\equiv\sup_{\mathcal{P}} L_{f;\mathcal{P}}\geq\sup_{\mathcal{P}_n} L_{f;\mathcal{P}_n}=\sup\{\frac{1}{2}(1-\frac{1}{n}):n\in\mathbb{N}\}=\frac{1}{2}$

only some partitions

$U_f=\inf_{\mathcal{P}} U_{f;\mathcal{P}}\leq\inf_{\mathcal{P}_n} U_{f;\mathcal{P}_n}=\inf\{\frac{1}{2}(1+\frac{1}{n}):n\in\mathbb{N}\}=\frac{1}{2}$

$\therefore\ \frac{1}{2}\le L_f\le U_f\le\frac{1}{2}\Rightarrow L_f=U_f=\frac{1}{2}$

Prp $\overset{\text{Dfn 3}}{\Longrightarrow} f:[0,1]\to\mathbb{R}$ is Riemann integrable, $\int_0^1 x\,dx=\frac{1}{2}$

Need to prove Prp: #2

Lemma (a) $\left(\inf_{x\in[a,b]} f(x)\right)\cdot(b-a)\le L_{f;\mathcal{P}}\le U_{f;\mathcal{P}}\le\left(\sup_{x\in[a,b]} f(x)\right)(b-a)$

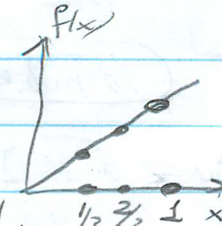
for every partition $\mathcal{P}$ of $[a,b]$

(b) If $\mathcal{P}'\supset\mathcal{P}$, then $L_{f;\mathcal{P}}\le L_{f;\mathcal{P}'}\le U_{f;\mathcal{P}'}\le U_{f;\mathcal{P}}$

Pf of (a): $L_{f;\mathcal{P}}\equiv\sum_{k=1}^{n}\left(\inf_{x\in[x_{k-1},x_k]} f(x)\right)(x_k-x_{k-1})$

$\inf_{x\in[a,b]} f(x)\le\ \le\sup_{x\in[x_{k-1},x_k]} f(x)\le\sup_{x\in[a,b]} f(x)$

Example 1: $f:[0,1]\to\mathbb{R}$, $f(x)=x$



$\mathcal{P}=\{a=0<\frac{1}{n}<\frac{2}{n}<\dots<\frac{n}{n}=1\}$

equal spacing $\Delta x=x_k-x_{k-1}=\frac{1}{n}$

$\Rightarrow L_{f;\mathcal{P}}=\sum_{k=1}^{n}\left(\inf_{x\in[\frac{k-1}{n},\frac{k}{n}]} f(x)\right)\frac{1}{n}=\frac{1}{n}\sum_{k=1}^{n}\frac{k-1}{n}=\frac{1}{n^2}\frac{(n-1)n}{2}=\frac{n-1}{2n}$

$U_{f;\mathcal{P}}=\sum_{k=1}^{\infty}\left(\sup_{x\in[\frac{k-1}{n},\frac{k}{n}]} f(x)\right)\frac{1}{n}=\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}=\frac{1}{n^2}\frac{n(n+1)}{2}=\frac{n+1}{2n}$

Note: $\lim_{n\to\infty}\frac{n-1}{2n}=\lim_{n\to\infty}\frac{n+1}{2}=\frac{1}{2}\quad\left(=\int_0^1 x\,dx\right)$

#4 (do not erase)

$f:[a,b]\to\mathbb{R}$ bounded, $a<b$

Dfn 2: $L_f\equiv\sup_{\mathcal{P}} L_{f;\mathcal{P}}$, $U_f\equiv\inf_{\mathcal{P}} U_{f;\mathcal{P}}$ — all partitions of $[a,b]$

Prp: $L_f\le U_f$

Dfn 3: Bounded $f:[a,b]\to\mathbb{R}$ is Riemann integrable if $L_f=U_f$. If so, $\int_a^b f\,dx\equiv L_f=U_f$

Example 2': $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$

$L_{f;\mathcal{P}}=0$ for every partition $\mathcal{P}$ of $[0,1]$ #3

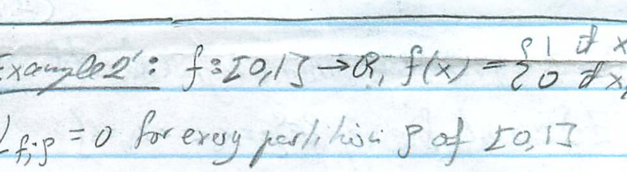
$U_{f;\mathcal{P}}=1$ — // —

$\Rightarrow L_f=\sup_{\mathcal{P}} L_{f;\mathcal{P}}=0$, $U_{f;\mathcal{P}}=\inf_{\mathcal{P}} U_{f;\mathcal{P}}=1$ — all partitions

$\Rightarrow L_f\neq U_f\overset{\text{Dfn 3}}{\Longrightarrow} f$ is not Riemann integrable on $[0,1]$

$\left(\inf_{x\in[a,b]} f(x)\right)\underbrace{\sum_{k=1}^{n}(x_k-x_{k-1})}_{b-a}\le L_{f;\mathcal{P}}\le U_{f;\mathcal{P}}\le\left(\sup_{x\in[a,b]} f(x)\right)\underbrace{\sum_{k=1}^{n}(x_k-x_{k-1})}_{b-a}$ ✓

Pf of (b): $\mathcal{P}'\equiv\{a=x_0'<x_1'<\dots<x_{n'}'=b\}\supset\{a=x_0<x_1<\dots<x_n=b\}=\mathcal{P}$



#3

$\Longrightarrow\{x_i=x_{k_i}'<x_k'<\dots<x_{k_{i+1}}'=x_{i+1}\}$ is a partition of $[x_i,x_{i+1}]$

(#2)

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(a) $\Rightarrow \left(\inf_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) \le \mathcal{L}_{f;\,\mathcal{P}'\cap[x_{i-1},x_i]} \le \mathcal{U}_{f;\,\mathcal{P}'\cap[x_{i-1},x_i]} \le \left(\sup_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})$

$\sum_i \Rightarrow \mathcal{L}_{f;\mathcal{P}} \le \mathcal{L}_{f;\mathcal{P}'} \le \mathcal{U}_{f;\mathcal{P}'} \le \mathcal{U}_{f;\mathcal{P}}$ ✓

(#2)

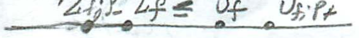
Pf of Prop: $\mathcal{L}_f \equiv \sup_{\mathcal{P}'} \mathcal{L}_{f;\mathcal{P}'} \overset{\text{Cor}}{\le} \inf_{\mathcal{P}} \mathcal{U}_{f;\mathcal{P}} \le \mathcal{U}_f$ (all partitions of $[a,b]$)

Thm 1: Bounded $f:[a,b]\to\mathbb{R}$ is Riemann integrable iff $\forall \varepsilon>0$, $\exists$ a partition $\mathcal{P}$ of $[a,b]$ s.t. $\mathcal{U}_{f;\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}} < \varepsilon$ (⊗)

Pf: Let $\varepsilon>0$. $\mathcal{L}_f = \sup_{\mathcal{P}_-} \mathcal{L}_{f;\mathcal{P}_-}$; $\mathcal{U}_f = \inf_{\mathcal{P}_+} \mathcal{U}_{f;\mathcal{P}_+}$

$\Rightarrow \exists$ partitions $\mathcal{P}_-, \mathcal{P}_+$ of $[a,b]$ s.t. $(0\le)\ \mathcal{L}_f - \mathcal{L}_{f;\mathcal{P}_-} < \varepsilon$

$(0\le)\ \mathcal{U}_{f;\mathcal{P}_+} - \mathcal{U}_f < \varepsilon$ (**)



(#4)

Thm 2: Bounded $f:[a,b]\to\mathbb{R}$ is integrable iff

(*) $\forall\varepsilon>0$, $\exists\delta>0$ s.t. for every $\mathcal{P}=\{a=x_0<\ldots<x_n=b\}$ with $|x_i - x_{i-1}|<\delta\ \forall i=1,\ldots,n$, $\mathcal{U}_{f;\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}} < \varepsilon$ ★

Pf: ε-δ condition $\Rightarrow$ ⊗ (just need one $\mathcal{P}$ so that ⊗ holds) $\overset{\text{Thm 1}}{\Longrightarrow}$ f is integrable

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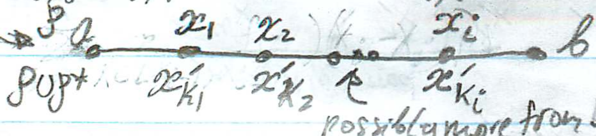
Lemma (b) $\Rightarrow$ $\mathcal{L}_{f;\mathcal{P}^*}, \mathcal{L}_{f;\mathcal{P}} \le \mathcal{L}_{f;\mathcal{P}^*\cup\mathcal{P}} \le \mathcal{U}_{f;\mathcal{P}^*\cup\mathcal{P}} \le \mathcal{U}_{f;\mathcal{P}^*}, \mathcal{U}_{f;\mathcal{P}}$

$+$ ⊛ $\Rightarrow$ $\mathcal{U}_{f;\mathcal{P}^*\cup\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}^*\cup\mathcal{P}} < \varepsilon$

Show $\mathcal{U}_{f;\mathcal{P}} - \mathcal{U}_{f;\mathcal{P}^*\cup\mathcal{P}} \le 2\varepsilon$, $\mathcal{L}_{f;\mathcal{P}^*\cup\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}} \le 2\varepsilon$

$\Rightarrow \mathcal{U}_{f;\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}} \le 4\varepsilon + \mathcal{U}_{f;\mathcal{P}^*\cup\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}^*\cup\mathcal{P}} < 5\varepsilon$ ✓



(#3)

Cor: $\mathcal{L}_{f;\mathcal{P}'} \le \mathcal{U}_{f;\mathcal{P}}$ $\forall$ partitions $\mathcal{P}, \mathcal{P}'$ of $[a,b]$

Pf: $\mathcal{P}\cup\mathcal{P}'$ = partition of $[a,b]$, combines all "break pts" of $\mathcal{P}$ and $\mathcal{P}'$

lemma (b) $\Rightarrow \mathcal{L}_{f;\mathcal{P}'} \le \mathcal{L}_{f;\mathcal{P}\cup\mathcal{P}'} \le \mathcal{U}_{f;\mathcal{P}\cup\mathcal{P}'} \le \mathcal{U}_{f;\mathcal{P}}$

$\Rightarrow \mathcal{L}_{f;\mathcal{P}'} \le \mathcal{U}_{f;\mathcal{P}}$

standard trick: $\mathcal{P}, \mathcal{P}' \rightsquigarrow$ common refinement $\mathcal{P}\cup\mathcal{P}'$

(do not erase Cor) (#3)

Lemma (b): $\mathcal{L}_{f;\mathcal{P}_-} \le \mathcal{L}_{f;\mathcal{P}_-\cup\mathcal{P}_+} \le \mathcal{U}_{f;\mathcal{P}_-\cup\mathcal{P}_+} \le \mathcal{U}_{f;\mathcal{P}_+}$

$\Rightarrow 0 \le \mathcal{L}_f - \mathcal{L}_{f;\mathcal{P}_-\cup\mathcal{P}_+} < \varepsilon$, $0 \le \mathcal{U}_{f;\mathcal{P}_-\cup\mathcal{P}_+} - \mathcal{U}_f < \varepsilon$

$0 \le (\mathcal{U}_{f;\mathcal{P}_-\cup\mathcal{P}_+} - \mathcal{L}_{f;\mathcal{P}_-\cup\mathcal{P}_+}) - (\mathcal{U}_f - \mathcal{L}_f) < 2\varepsilon$

f integrable $\Leftrightarrow \mathcal{U}_f = \mathcal{L}_f \Rightarrow 0 \le \mathcal{U}_{f;\mathcal{P}_-\cup\mathcal{P}_+} - \mathcal{L}_{f;\mathcal{P}_-\cup\mathcal{P}_+} < 2\varepsilon \Rightarrow$ ⊗

Suppose ⊗ holds $\Rightarrow \mathcal{L}_{f;\mathcal{P}} \le \mathcal{L}_f \le \mathcal{U}_f \le \mathcal{U}_{f;\mathcal{P}} < \mathcal{L}_{f;\mathcal{P}} + \varepsilon$

$\Longrightarrow 0 \le \mathcal{U}_f - \mathcal{L}_f < \varepsilon\ \forall\varepsilon>0 \overset{\text{Prop}}{\Longrightarrow} \mathcal{U}_f = \mathcal{L}_f \Leftrightarrow f$ integrable

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(#1)

Suppose f is integrable and $\varepsilon>0$. Let $M = \sup|f| \in \mathbb{R}^+$

Thm 1 $\Rightarrow \exists$ partition $\mathcal{P}^* = \{a = x_0^* < \ldots < x_{n^*}^* = b\}$ of $[a,b]$ s.t.

$\mathcal{U}_{f;\mathcal{P}^*} - \mathcal{L}_{f;\mathcal{P}^*} < \varepsilon$ ⊛

Take $\delta = \frac{\varepsilon}{M n^*}$. Given $\mathcal{P} = \{a = x_0 < \ldots < x_n = b\}$ with $|x_i - x_{i-1}| < \delta\ \forall i=1,\ldots,n$, show $\mathcal{U}_{f;\mathcal{P}} - \mathcal{L}_{f;\mathcal{P}} < 5\varepsilon$

$\mathcal{U}_{f;\mathcal{P}} - \mathcal{U}_{f;\mathcal{P}\cup\mathcal{P}^*} = \sum_{i=1}^{n}\left(\sup_{x\in[x_{i-1},x_i]} f(x)\cdot(x_i - x_{i-1}) - \sum_{k=k_{i-1}+1}^{k_i} \sup_{x\in[x'_{k-1},x'_k]} f(x)\,(x'_k - x'_{k-1})\right)$ (at most n^* of these are different)

$\le M(x_i - x_{i-1}) + M\sum_{k=k_{i-1}+1}^{k_i}(x'_k - x'_{k-1})$, where $\sum_{k=k_{i-1}+1}^{k_i}(x'_k - x'_{k-1}) = x_i - x_{i-1}$

$\le 2M\delta \le \frac{2\varepsilon}{n^*}$

$\sum_i \Longrightarrow \mathcal{U}_{f;\mathcal{P}} - \mathcal{U}_{f;\mathcal{P}\cup\mathcal{P}^*} \le \frac{2\varepsilon}{n^*}\cdot n^*$