Pf of (c): $B_n \subset X$ nowhere dense $\Rightarrow X - \overline{B}_n \subset X$ is dense and open (#2)

Thm 3 $\Rightarrow \bigcap_{n=1}^{\infty}(X - \overline{B}_n) \subset X$ is dense

$X - \bigcup_{n=1}^{\infty} B_n \supset X - \bigcup_{n=1}^{\infty} \overline{B}_n \Rightarrow X - \bigcup_{n=1}^{\infty} B_n$ dense

Example: $q \in \mathbb{R} \Rightarrow \{q\}$ nowhere dense in $\mathbb{R}$ b/c $\mathbb{R} - \{q\} \subset \mathbb{R}$ is dense

Crl 3c $\Rightarrow \mathbb{R} - \bigcup_{q \in \mathbb{Q}} \{q\} \subset \mathbb{R}$ is dense ✓

(#2)

X perfect $\Rightarrow \{x\} \subset X$ nowhere dense for all $x \in X$

X complete $\overset{Crl 2(b)}{\Longrightarrow} X \neq$ countable union of $\{x\}$'s

Crl 4: No nondegenerate interval $I \subset \mathbb{R}$ can be written as $\amalg$ of two or more closed nondegenerate intervals

$I \subset \mathbb{R}$ nondegenerate if len$(I) > 0$

i.e. $I = [a,b]$ $a<b$, $(-\infty, b]$, $[a, \infty)$, $(-\infty, \infty)$

---

(3) Can assume I contains no endpoints a, b.

E.g. if $I = [a, b]$, $a_m = a$ for some m

$\Rightarrow (b_m, b] = I - [a_m, b_m] = \coprod_{n \neq m} [a_n, b_n]$

(4) Assuming I contains no endpoints a, b, take

$K = \{a_n, b_n : n \in \mathbb{N}\} \cup \{a, b\} = I - \bigcup_{n=1}^{\infty} (a_n, b_n)$

if $a, b \in \mathbb{R}$ $\quad = \mathbb{R} - (-\infty, a) - (b, \infty) - \bigcup_{n=1}^{\infty} (a_n, b_n)$

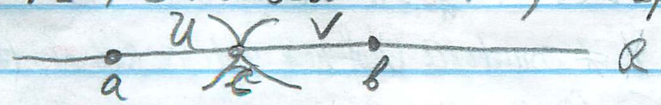
countable, closed, perfect $\Rightarrow$ contradicts Crl 3

Prp: The connected subsets of $\mathbb{R}$ are the intervals I

i.e. $A \subset \mathbb{R}$ connected iff $\forall a, b \in A$ and $a < c < b$, $c \in A$

Pf: (i) If A is not interval, then A is disconnected

$\Rightarrow \exists a, b \in A$ and $c \in \mathbb{R}$ with $a < c < b$, but $c \notin A$



$\Rightarrow A \subset (-\infty, c) \cup (c, \infty) = U \cup V$, $U, V \subset \mathbb{R}$ open

$a \in A \cap U \neq \emptyset$, $b \in A \cap V \neq \emptyset$, $U \cap V = \emptyset$

(do not erase)

(#3)

Dfn 2: (X,d) is perfect if it is complete and $\forall x \in X$; $\exists$ a sequence $(x_n)_n$ in $X - \{x\}$ s.t. $x_n \to x$

E.g. $[a,b]$ with $a<b$ is perfect, but $\mathbb{Z}, \mathbb{Q} \subset \mathbb{R}$ are not

Crl 3: If (X,d) is perfect, then X is uncountable

Pf: $\exists$ sequence $(x_n)_n$ in $X - \{x\}$ s.t. $x_n \to x$

$\Rightarrow X - \{x\} = X - \overline{\{x\}}$ is dense in X, i.e. $\overline{X - \{x\}} = X$

~~X perfect ⇒ {x} nowhere dense for all~~

Pf: (1) $I \neq [a_1, b_1] \amalg [a_2, b_2] \amalg \dots \amalg [a_k, b_k]$ with $k \geq 2$

o/w can assume $a_1 < b_1 < a_2 < b_2 < \dots$

and thus missing (b_1, a_2), etc.

(2) Suppose $I = \amalg [a_n, b_n]$ ∞-many of these must be countable b/c $[a_n, b_n] \cap \mathbb{Q} \neq \emptyset$ $\forall n$

Connectedness $\rightsquigarrow$ Intermediate Value Thm

Dfn: (X,d) metric space

(a) $A \subset X$ is disconnected if $\exists U, V \subset X$ open s.t.

$A \cap U, A \cap V \neq \emptyset$, $U \cap V = \emptyset$, and $A \subset U \cup V$

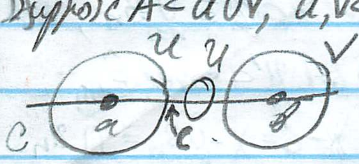
(b) $A \subset X$ is connected if A is not disconnected

U (discon) (nected) V $\qquad$ (conne)(cted) $\leftarrow U \cap V \neq \emptyset$

(ii) If A is disconnected, then A is not an interval

Suppose $A \subset U \cup V$, $U, V \subset \mathbb{R}$ open, $A \cap U, B \cap V \neq \emptyset$

$a < b$



$c = \sup\{t \in \mathbb{R} : [a, t] \subset U\} \in [a, b]$

$a \leq$ $\quad t < b$

$c \notin U$ b/c U is open

$c \in V$ b/c V is open

$\Rightarrow c \notin A$

$\Rightarrow A$ is not interval

MAT 320

⟹ (color chalk) ⟸

3/28/19

(do not erase)

#4

#1

Last time: Properties of Compactness

<u>Thm 1:</u> (X,d) metric space. $A \subset X$ cmpt iff every sequence $(x_n)_n$ in A has a subseq. $(x_{n_k})_k$ converging in A (sequentially cmpt)

<u>Thm 2:</u> (X,d_X), (Y,d_Y) metric space. If $A \subset X$ and $B \subset Y$ cmpt, then $A \times B \subset X \times Y$ is cmpt w.r.t. max/square, round, sum/rhombus metric on $X \times Y$

<u>E.g.</u> $\mathbb{Q}$ is dense in $\mathbb{R}$, but not open and not countable $\cap$ of open subsets of $\mathbb{R}$ (HW 7)

But $\mathbb{R} - \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} - \{q\})$

For Thm 3, need (X,d) complete.

<u>E.g.</u> $A \equiv \bigcap_{q \in \mathbb{Q}} (\mathbb{Q} - \{q\}) = \emptyset$ not dense in $\mathbb{Q}$ (open in $\mathbb{Q}$, but not in $\mathbb{R}$)

Today: Properties of Completeness

<u>Dfn 9:</u> (X,d) is <u>complete</u> if every Cauchy sequence $(x_n)_n$ converges

<u>Thm 3:</u> (X,d) complete metric space

If $\mathcal{U}_1, \mathcal{U}_2, \ldots \subset X$ are open and dense in X, then $A \equiv \bigcap_{n=1}^{\infty} \mathcal{U}_n \subset X$ is also dense

$A \subset X$ <u>dense</u> if $\overline{A} = X$

$\iff \forall \mathcal{U} \subset X$ open with $\mathcal{U} \neq \emptyset$, $A \cap \mathcal{U} \neq \emptyset$

<u>Pf:</u> Let $\mathcal{U} \subset X$ be open with $\mathcal{U} \neq \emptyset$. Show $A \cap \mathcal{U} \neq \emptyset$

$\iff \exists x \in X$ s.t. $x \in \mathcal{U}_n \cap \mathcal{U}$ $\forall n \in \mathbb{N}$

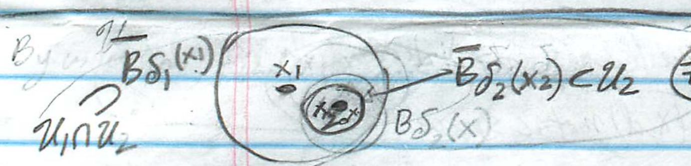
$\mathcal{U}_1 \subset X$ dense, $\mathcal{U} \neq \emptyset$ open $\Rightarrow \exists x_1 \in \mathcal{U}_1 \cap \mathcal{U}$

$\mathcal{U}_1 \cap \mathcal{U}$ open $\Rightarrow \exists \delta_1 \in (0,1)$ s.t. $\overline{B}_{\delta_1}(x_1) \subset \mathcal{U}_1 \cap \mathcal{U}$

closed ball $\to \{x' \in X : d(x,x') \le \delta_1\} \subset B_{2\delta_1}(x_1)$

$\mathcal{U}_2 \subset X$ dense, $B_{\delta_1}(x_1) \neq \emptyset$ open $\Rightarrow \exists x_2 \in \mathcal{U}_2 \cap B_{\delta_1}(x_1)$

$\mathcal{U}_2 \cap B_{\delta_1}(x_1)$ open $\Rightarrow \exists \delta_2 \in (0, 1/2)$ s.t. $\overline{B}_{\delta_2}(x_2) \subset \mathcal{U}_2 \cap B_{\delta_1}(x_1)$



#4

By induction, get $x_1, x_2, \ldots \in \mathcal{U}$ and $\delta_1, \delta_2, \ldots \in \mathbb{R}^+$ s.t. $\delta_n \le \frac{1}{2^{n-1}}$, $\overline{B}_{\delta_1}(x_1) \subset \mathcal{U}$, $\overline{B}_{\delta_{n+1}}(x_{n+1}) \subset \mathcal{U}_n \cap B_{\delta_n}(x_n)$

$x_{n+1} \in B_{\delta}(x_n) \Rightarrow d(x_n, x_{n+1}) < \delta_n \le \frac{1}{2^{n-1}}$

$\Rightarrow \forall m \le n$, $d(x_m, x_n) \le \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \le \sum_{k=m}^{n-1} \frac{1}{2^{k-1}} < \frac{1}{2^{m-2}}$

#2

$\Rightarrow d(x_n, x_{n'}) \le 2 \cdot \frac{1}{2^{m-2}}$ $\forall n, n' \ge m$

$\Rightarrow (x_n)_n$ is Cauchy $\Rightarrow$ converges to some $x \in X$

$x_{n+1} \in \overline{B}_{\delta_n}(x_n) \subset \overline{B}_{\delta_{n-1}}(x_{n-1}) \subset \ldots \subset \overline{B}_{\delta_1}(x_1) \subset \mathcal{U}$

$\Rightarrow x_n \in \overline{B}_{\delta_m}(x_m)$ $\forall n \ge m$

$\searrow x$ closed $\Rightarrow x \in \overline{B}_{\delta_m}(x_m) \subset \mathcal{U}_m \cap \mathcal{U}$

$\Rightarrow A \cap \mathcal{U} \equiv \left(\bigcap_{n=1}^{\infty} \mathcal{U}_n\right) \cap \mathcal{U} \neq \emptyset$ ✓

#3

<u>Crl 1:</u> (X,d) complete metric space, $B_1, B_2, \ldots \subset X$ closed

If $\exists \mathcal{U} \subset X$ open s.t. $\mathcal{U} \neq \emptyset$ and $\mathcal{U} \subset C \equiv \bigcup_{n=1}^{\infty} B_n$,

then $\exists \mathcal{U}' \subset X$ open s.t. $\mathcal{U}' \neq \emptyset$ and $\mathcal{U}' \subset$ some B_n

<u>Pf:</u> Take $\mathcal{U}_n = X - B_n$ open $\Rightarrow X - C = X - \bigcup_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \mathcal{U}_n$

$\mathcal{U} \subset C \iff \mathcal{U} \cap (X - C) = \emptyset$, $\mathcal{U} \subset X$ open, $\mathcal{U} \neq \emptyset$ $\Longrightarrow X - C \subset X$ is not dense

$\Downarrow$ Thm 3

$\exists \mathcal{U}' \subset X$ open with $\mathcal{U}' \neq \emptyset$ s.t. $\mathcal{U}' \cap \mathcal{U}_n = \emptyset$ $\Longleftarrow$ $\mathcal{U}_n \subset X$ not dense for some n

$\mathcal{U}' \subset B_n$

#4

<u>Dfn:</u> $B \subset X$ is <u>nowhere dense</u> if $X - \overline{B} \subset X$ is dense

<u>Crl 2:</u> (X,d) complete metric space

(a) If $B_1, B_2, \ldots \subset X$ are nowhere dense, then $X - \bigcup_{n=1}^{\infty} B_n \subset X$ is dense

(b) $X \neq$ countable $\cup$ of nowhere dense subsets (if $X \neq \emptyset$)

(a) $\Rightarrow$ (b) b/c $B_1, B_2, \ldots \subset X$ nowhere dense $\Rightarrow$ $X - \bigcup_{n=1}^{\infty} B_n \subset X$ is dense $\Rightarrow \neq \emptyset$

MAT 320: Introduction to Analysis, Spring 2019
Baire Spaces

Let (X, d) be a metric space and $A \subset X$. The interior of A, or Int A, is the largest open subset of X contained in A; this is the union of all open subsets contained in A. The interior of A is empty if and only if no nonempty open subset U of X is contained in A, i.e. every nonempty open subset U of X intersects $X - A$. The last condition means that $X - A$ is dense in X.

A metric space (X, d) is called Baire if the intersection

$$\bigcap_{n=1}^{\infty} U_n \subset X$$

is dense in (X, d) for every sequence $U_1, U_2, \ldots \subset X$ of dense open subsets of (X, d); this is Ross's property 21.7a. This is equivalent to the condition that the union

$$\bigcup_{n=1}^{\infty} F_n \subset X$$

of closed sets $F_1, F_2, \ldots \subset X$ with empty interiors has empty interior; this is Ross's property 21.7b. This condition is in turn equivalent to the condition that no nonempty open subset $W \subset X$ is a countable union of nowhere dense subsets of X, i.e. every open subset $W \subset X$ is of Category 2.

The equivalence of the first two conditions above is obtained as follows. Let $U_1, U_2, \ldots \subset X$ be any sequence of subsets and $B_n = X - U_n$. Each set U_n is open (resp. dense) in X if and only if each set B_n is closed (resp. has empty interior in X); the second equivalence is by the first paragraph above. The intersection of the sets U_n is dense in X if and only if its complement

$$X - \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X - U_n) = \bigcup_{n=1}^{\infty} B_n$$

has empty interior.

Baire Category Theorem. A complete metric space is a Baire space.

This implies that a complete metric space (X, d) is of Category 2 in itself. This is the statement of Ross's Theorem 21.8, i.e. this theorem is a corollary of the usual formulation of Baire Category Theorem, which is much weaker than the theorem itself. For example, let (X, d) be a metric space consisting of $\mathbb{Q}$ and another isolated point p^*, e.g.

$$X = \mathbb{Q} \sqcup \{p^*\}, \qquad d(x, x') = \begin{cases} |x - x'|, & \text{if } x, x' \in \mathbb{Q}; \\ 1, & \text{if } x \neq x',\ p^* \in \{x, x'\}; \\ 0, & \text{if } x = x'. \end{cases}$$

A nowhere dense subset F in this space cannot contain p^* (because $\{p^*\}$ is open in this metric space). Thus, (X, d) is of Category 2. However, the open subset $\mathbb{Q}$ of X is not of the second category, since it is a countable union of its own points, which are nowhere dense in $\mathbb{Q}$ and X.

21.11 Theorem.
No nondegenerate interval I in $\mathbb{R}$ can be written as the union of two or more disjoint nondegenerate closed intervals. (An interval is "nondegenerate" if it has more than one point.)

We first note that I cannot be written as the union of finitely many disjoint nondegenerate closed intervals,

$$I = [a_1, b_1] \sqcup [a_2, b_2] \sqcup \ldots \sqcup [a_k, a_k], \tag{1}$$

with $k \geq 2$. If it were possible to do so, we could assume that

$$a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k.$$

The open interval interval (b_1, a_2) would then be missing from the right side of (1), and so this union would not be an interval. The same reasoning would apply if the first interval in (1) were of the form $(-\infty, b_1]$ or the last were of the form $[a_k, +\infty)$.

It remains to consider a decomposition of I into infinitely many disjoint nondegenerate closed intervals. Since every such interval contains a rational number, the number of these intervals must be countable. Thus, suppose that

$$I = \bigsqcup_{n \in \mathbb{N}} [a_n, b_n]. \tag{2}$$

Let $a = \inf I \in \mathbb{R} \sqcup \{-\infty\}$ and $b = \sup I \in \mathbb{R} \sqcup \{+\infty\}$. If $a \in I$, then $a_m = a$ for some $m \in \mathbb{N}$. Dropping the interval $[a_m, b_m]$ from both sides of (2), we would get a similar decomposition for an interval without the left endpoint. By the same reasoning, we can drop the right endpoint from I. We can thus assume that $a, b \notin I$, i.e. I is open in $\mathbb{R}$.

Let E be the set of the endpoints a_n, b_n of all intervals in (2) along with a if $a \in \mathbb{R}$ and b if $b \in \mathbb{R}$. Since

$$E = \mathbb{R} - (-\infty, a) - (b, +\infty) - \bigcup_{n=1}^{\infty} (a_n, b_n),$$

the subset E is closed in $\mathbb{R}$. We show that it is also perfect, i.e. for every $x \in E$ and $\delta \in \mathbb{R}^+$ the open ball $(x-\delta, x+\delta)$ around x in $\mathbb{R}$ contains a point $x' \in E$ different from x. Since $x \in I$ and $I \subset \mathbb{R}$ is open,

$$I \cap (x-\delta, x), I \cap (x, x+\delta) \neq \emptyset.$$

By (2), there thus exist $k, m \in \mathbb{N}$ such that

$$[a_k, b_k] \cap (x-\delta, x), [a_m, b_m] \cap (x, x+\delta) \neq \emptyset.$$

If $x = a_n$ for some n, then $x \notin [a_k, b_k]$ and thus $x' \equiv b_k \in E \cap (x-\delta, x)$. If $x = b_n$ for some n, then $x \notin [a_m, b_m]$ and thus $x' \equiv a_m \in E \cap (x, x+\delta)$. In either case, we find a point $x' \in E \cap (x-\delta, x+\delta)$ different from x. The same reasoning applies to decompositions as in (2) containing an interval form $(-\infty, b_n]$ or an interval of the form $[a_n, +\infty)$.

In summary, we find that $E \subset \mathbb{R}$ is a perfect subset. Since E is countable, this contradicts the conclusion of Discussion 21.10 and so a decomposition (2) does not exist.

21.11 Theorem.
No interval I in $\mathbb{R}$ can be written as the union of two or more disjoint nondegenerate closed intervals. (An interval is "nondegenerate" if it has more than one point.)

A closed nondegenerate interval is all of $\mathbb{R}$, or a closed half-infinite interval $(-\infty, b]$ with $b \in \mathbb{R}$ or $[a, +\infty)$ with $a \in \mathbb{R}$, or a closed bounded interval with $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. Suppose there is a decomposition of $\mathbb{R}$ of the form

$$\mathbb{R} = (-\infty, b] \sqcup [a, +\infty) \bigsqcup_{\alpha \in \mathcal{A}} [a_\alpha, b_\alpha] \tag{1}$$

for some indexing set $\mathcal{A}$ and $a, b, a_\alpha, b_\alpha \in \mathbb{R}$ with $a_\alpha < b_\alpha$. Since every interval $[a_\alpha, b_\alpha]$ contains a rational number, the set $\mathcal{A}$ is at most countable. Let E be the set of the endpoints a, b, a_α, b_α of all intervals in (1). Since

$$E = \mathbb{R} - (-\infty, b) - (a, +\infty) - \bigcup_{\alpha \in \mathcal{A}} (a_\alpha, b_\alpha),$$

the subset E is closed in $\mathbb{R}$. We show in the next paragraph that E is also perfect. Since E is nonempty, but at most countable, this contradicts the conclusion of Discussion 21.10, and so a decomposition (1) does not exist.

We now show that the set E of endpoints of the intervals in (1) is perfect, i.e. for every $x \in E$ and $\delta \in \mathbb{R}^+$ the open ball $(x-\delta, x+\delta)$ around x in $\mathbb{R}$ contains a point $x' \in E$ different from x. By (1), there exist $\beta, \gamma \in \mathcal{A}$ such that

$$[a_\beta, b_\beta] \cap (x-\delta, x), [a_\gamma, b_\gamma] \cap (x, x+\delta) \neq \emptyset.$$

If x is a left endpoint, i.e. $x = a$ or $x = a_\alpha$ for some $\alpha \in \mathcal{A}$, then $x \notin [a_\beta, b_\beta]$ and thus

$$x' \equiv b_\beta \in E \cap (x-\delta, x).$$

If x is a right endpoint, i.e. $x = b$ or $x = b_\alpha$ for some $\alpha \in \mathcal{A}$, then $x \notin [a_\gamma, b_\gamma]$ and thus

$$x' \equiv a_\gamma \in E \cap (x, x+\delta).$$

In either case, we find a point $x' \in E \cap (x-\delta, x+\delta)$ different from x.

Suppose I is any interval decomposed as the union of two or more disjoint nondegenerate closed intervals. Let b_β be the right endpoint of an interval in the decomposition and a_γ be the left endpoint of another interval in the decomposition so that $b_\beta < a_\gamma$. The intervals $[a_\alpha, b_\alpha]$ in the decomposition of I so that $a_\alpha, b_\alpha \in (b_\beta, a_\gamma)$ then decompose the open interval (b_β, a_γ) into disjoint closed intervals. Adding $(-\infty, b_\beta]$ and $[a_\alpha, +\infty)$ to this decomposition of (b_β, a_γ), we obtain a decomposition of $\mathbb{R}$ as in (1). Since the latter does not exist, we conclude that no interval I in $\mathbb{R}$ can be written as the union of two or more disjoint nondegenerate closed intervals.