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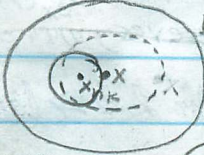
Before break:  $(X, d)$  metric space (#1)  
 $\mathcal{U} \subset X$  open if  $\forall x \in \mathcal{U}, \exists \delta_x \in \mathbb{R}^+$  s.t.  $B_{\delta_x}(x) \subset \mathcal{U}$   
 $A \subset X$  compact if every open cover  $\mathcal{C}$  of  $A$   
 contains a finite subcollection  $\mathcal{C}'$  covering  $A$   
 $(x_n)_n$  converges to  $x \in X$  if  $\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$  s.t.  $x_n \in U \forall n \geq N$

$\Rightarrow \exists \epsilon_k \rightarrow 0$  and  $U \subset X$  open with  $x \in U$ ;  
 $\exists K \in \mathbb{Z}^+$  s.t.  $x_k \in U \forall k \geq N \Leftrightarrow n_k \in \mathbb{N} \forall k \geq N$   
 $\Rightarrow n_k < n_{k+1} \dots \Rightarrow$  get to manage element of  $\mathbb{N}(U)$   
 $n_k = \min \mathbb{N}(B_{1/n_k}(x)) \in \mathbb{N}$   
 non-empty subset of  $\mathbb{N}$   
 $n_{k+1} = \min \{ n \in \mathbb{N} \mid B_{1/n}(x) \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset \}$   
 $d(x_{n_{k+1}}, x) < 1/n_{k+1} \forall k \geq N \Rightarrow (x_{n_k})_k \rightarrow x$

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Lemma 2 (Telesque Number Lemma)  $(X, d)$  = metric space  
 $A \subset X$  is sequentially compact and  $\mathcal{C}$  an open cover of  $A$ ,  
 $\exists \delta \in \mathbb{R}^+$  s.t.  $\forall x \in A, \exists U_x \in \mathcal{C}$  with  $B_\delta(x) \subset U_x$   
 $U_1 \quad U_2$  : same  $\delta$  for all  $x \in A$ !

choose  $k \geq N$  s.t.  $1/n_k \leq \delta_x$   
 $d(x, x_{n_k}) < \delta_x$   
 $d(x_{n_k}, x') < 1/n_k \leq \delta_x \forall x' \in B_{1/n_k}(x_{n_k})$   
 $\Rightarrow d(x, x') < 2\delta_x \forall x' \in B_{1/n_k}(x_{n_k})$   
 $\Rightarrow B_{1/n_k}(x_{n_k}) \subset B_{2\delta_x}(x) \subset U_x \in \mathcal{C}$   
 Contradiction b/c  $B_{1/n_k}(x_{n_k}) \not\subset U \forall U \in \mathcal{C}$



Thm:  $(X, d)$  metric space,  $A \subset X$  is compact iff every sequence  $(x_n)_n$  in  $A$  has a subsequence convergent in  $A$ .  
 $A \subset X$  is sequentially compact

Lemma 1:  $(x_n)_n$  has a subsequence  $(x_{n_k})_k$  convergent to  $x \in X$   
 iff  $\forall U \subset X$  open with  $x \in U, |\{n \in \mathbb{N} : x_n \in U\}| = \infty$

Pf of Thm  $\Rightarrow$ : Suppose  $(x_n)_n$  is a sequence in  $A \subset X$  and contains no subsequence  $(x_{n_k})_k$  convergent to any  $x \in X$   
 $\Rightarrow \forall x \in A, \exists U_x \subset X$  open with  $x \in U_x$  and  $|\mathbb{N}(U_x)| < \infty$   
 $\Rightarrow \mathcal{C} = \{U_x : x \in A\}$  is an open cover of  $A$   $\{n \in \mathbb{N} : x_n \in U_x\}$   
 $U_x \subset X$  open  $\forall U_x \in \mathcal{C}, A \subset \bigcup_{U_x \in \mathcal{C}} U_x$   
 $A$  compact  $\Rightarrow \exists \mathcal{C}' \subset \mathcal{C}$  finite s.t.  $A \subset \bigcup_{U_x \in \mathcal{C}'} U_x$   
 $\Rightarrow \mathbb{N} = \{n \in \mathbb{N} : x_n \in \bigcup_{U_x \in \mathcal{C}'} U_x\} = \bigcup_{U_x \in \mathcal{C}'} \mathbb{N}(U_x)$  finite  $\times$

Pf: Suppose not. Then for each  $n \in \mathbb{N}$   
 $\exists x_n \in A$  s.t.  $B_{1/n}(x_n) \not\subset U \forall U \in \mathcal{C}$   
 $A$  seq. compact  $\Rightarrow \exists$  subseq.  $(x_{n_k})_k \rightarrow$  some  $x \in A$   
 $\mathcal{C}$  covers  $A \Rightarrow \exists U_x \in \mathcal{C}$  s.t.  $x \in U_x$   
 $U_x \subset X$  open  $\Rightarrow \exists \delta_x \in \mathbb{R}^+$  s.t.  $B_{\delta_x}(x) \subset U_x$   
 $(x_{n_k})_k \rightarrow x \Rightarrow \exists N \in \mathbb{Z}^+$  s.t.  $d(x, x_{n_k}) < \delta_x \forall k \geq N$

Pf of Thm  $\Leftarrow$ : Let  $\mathcal{C}$  be an open cover of  $A$ .  
 Suppose  $\mathcal{C}$  contains no finite subcollection  $\mathcal{C}'$  covering  $A$ .  
 Lemma 2  $\Rightarrow \exists \delta \in \mathbb{R}^+$  s.t.  $\forall x \in A, \exists U_x \in \mathcal{C}$  with  $B_\delta(x) \subset U_x$   
 Take  $x_1 \in A$  any and  $x_2, x_3, \dots \in A$  inductively s.t.  
 $x_{n+1} \notin B_\delta(x_1) \cup \dots \cup B_\delta(x_n) \subset U_{x_1} \cup \dots \cup U_{x_n}$   
 $\leftarrow$  exists b/c  $\mathcal{C} = \{U_{x_1}, U_{x_2}, \dots\} \subset \mathcal{C}$  does not cover  $A$

$x_n \notin B_\delta(x_i) \nRightarrow d(x_i, x_n) \geq \delta \forall i \neq n$  (#4)  
 $\Rightarrow$  no subseq.  $(x_{n_k})_k$  of  $(x_n)_n$  is Cauchy  
 $\Rightarrow$  no subseq. converges in  $A$  (or  $X$ )  
 Contradicts to  $A \subset X$  sequentially compact

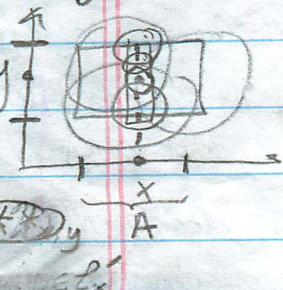
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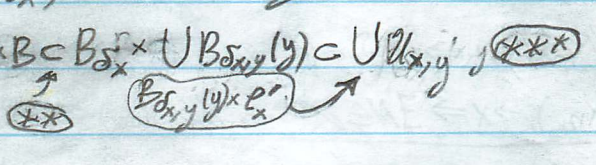
HW6, Problem 6-a: each  $d_i$  is a metric on  $X \times Y$   
 any two of these are uniformly equivalent  
 Problem F  $\Rightarrow$   $W \subset X \times Y$  is  $d_i$ -open iff  $d_j$ -open  
 $C \subset X \times Y$  is  $d_i$ -compact iff  $d_j$ -compact

Thm 2:  $(X, d_x), (Y, d_y)$  metric space. If  $A \subset X$  and  $B \subset Y$  compact  
 then  $A \times B \subset X \times Y$  is compact w.r.t. metric  

$$d_i((x, y), (x', y')) = \begin{cases} \max(d_x(x, x'), d_y(y, y')) \\ (d_x(x, x')^2 + d_y(y, y')^2)^{1/2} \\ d_x(x, x') + d_y(y, y') \end{cases}$$

so enough to prove Thm 2 for  $d = d_1$   
 $\rightarrow B_{\delta_1}(x, y) = B_{\delta_1}(x) \times B_{\delta_1}(y)$   
 open balls in  $(X \times Y, d_1)$  are "squares"

Prf of Thm 2: Let  $\mathcal{C}$  be an open cover of  $A \times B$   
 $\Rightarrow \forall x \in A, y \in B \exists \delta_{x,y} \in \mathbb{R}^+$  s.t.  $B_{\delta_{x,y}}(x, y) \subset \text{some } \mathcal{U}_{x,y} \in \mathcal{C}$   
 $\Rightarrow \forall x \in X, \mathcal{C}_x = \{B_{\delta_{x,y}}(y) : y \in B\}$   
 is open cover of  $B$   
 $B$  compact  $\Rightarrow \exists$  finite  $\mathcal{C}'_x \subset \mathcal{C}_x$   
 s.t.  $B \subset \bigcup B_{\delta_{x,y}}(y) \subset \bigcup_{y \in \mathcal{C}'_x} B_{\delta_{x,y}}(y) \subset \mathcal{C}'_x$   


Let  $\delta_x = \min \{ \delta_{x,y} : B_{\delta_{x,y}}(y) \in \mathcal{C}'_x \} \in \mathbb{R}^+$  (#3)  
 finite subset of  $\mathbb{R}^+$   
 $\Rightarrow B_{\delta_x}(x) \times B_{\delta_x}(y) \subset B_{\delta_x}(x) \times B_{\delta_x}(y) = B_{\delta_x}(x, y) \in \mathcal{U}_{x,y}$   
 $\forall y \in \mathcal{C}'_x, x \in X$  square metric  
 $\Rightarrow B_{\delta_x}(x) \times B \subset B_{\delta_x}(x) \times \bigcup B_{\delta_x}(y) \subset \bigcup \mathcal{U}_{x,y}$  (\*\*\*)  


$\mathcal{C}_A = \{ B_{\delta_x}(x) : x \in A \}$  is open cover of  $A$   
 $A$  compact  $\Rightarrow \exists$  finite  $\mathcal{C}'_A \subset \mathcal{C}_A$  s.t.  $A \subset \bigcup_{x \in \mathcal{C}'_A} B_{\delta_x}(x)$   
 $\Rightarrow A \times B \subset \bigcup_{x \in \mathcal{C}'_A} B_{\delta_x}(x) \times B \subset \bigcup_{x \in \mathcal{C}'_A} \bigcup_{y \in \mathcal{C}'_x} \mathcal{U}_{x,y}$   
 $\therefore \mathcal{C} = \bigcup_{x \in \mathcal{C}'_A} \mathcal{C}'_x \subset \mathcal{C}$  finite, covers  $A \times B$   
 $\Rightarrow$  every open cover of  $A \times B$  has finite subcover  $\checkmark$

HW6, §13.12  $B \subset A \subset X, A$  compact,  $B$  closed  $\Rightarrow B$  compact  
 Recitations yesterday and tomorrow:  $(X, d)$  metric spaces  
 (i)  $A \subset X$  compact  $\Rightarrow A$  bounded, closed, complete w.r.t.  
 (ii) Completeness Axiom for  $\mathbb{R} \Rightarrow$  Bolzano-Weierstrass  
 MAT 319  $\rightarrow [a, b] \subset \mathbb{R}$  is compact  $\xleftarrow{\text{Thm 1}}$   
 every bounded seq. in  $\mathbb{R}$  has a convergent subseq.

+ Thm 2  $\Rightarrow$  (ii) Heine-Borel Thm for  $(\mathbb{R}^n, d_{\text{eucl}})$   
 square/round/sum metric  
 $A \subset \mathbb{R}^n$  is  $d_{\text{eucl}}$  compact iff  $A$  is  $d_{\text{eucl}}$ -closed and  $d_{\text{eucl}}$ -bounded  
 Not true for subsets  $A$  of other metric spaces  $(X, d)$ ,  
 even complete ones: e.g. HW6 §13.3ac, 13.5

# J. Munkres, Topology

name. It still has not found a name on which everyone agrees. On historical grounds, some call it “Fréchet compactness”; others call it the “Bolzano-Weierstrass property.” We have invented the term “limit point compactness.” It seems as good a term as any; at least it describes what the property is about.

**Theorem 28.1.** *Compactness implies limit point compactness, but not conversely.*

*Proof.* Let  $X$  be a compact space. Given a subset  $A$  of  $X$ , we wish to prove that if  $A$  is infinite, then  $A$  has a limit point. We prove the contrapositive—if  $A$  has no limit point, then  $A$  must be finite.

So suppose  $A$  has no limit point. Then  $A$  contains all its limit points, so that  $A$  is closed. Furthermore, for each  $a \in A$  we can choose a neighborhood  $U_a$  of  $a$  such that  $U_a$  intersects  $A$  in the point  $a$  alone. The space  $X$  is covered by the open set  $X - A$  and the open sets  $U_a$ ; being compact, it can be covered by finitely many of these sets. Since  $X - A$  does not intersect  $A$ , and each set  $U_a$  contains only one point of  $A$ , the set  $A$  must be finite. ■

**EXAMPLE 1.** Let  $Y$  consist of two points; give  $Y$  the topology consisting of  $Y$  and the empty set. Then the space  $X = \mathbb{Z}_+ \times Y$  is limit point compact, for every nonempty subset of  $X$  has a limit point. It is not compact, for the covering of  $X$  by the open sets  $U_n = \{n\} \times Y$  has no finite subcollection covering  $X$ .

**EXAMPLE 2.** Here is a less trivial example. Consider the minimal uncountable well-ordered set  $S_\Omega$ , in the order topology. The space  $S_\Omega$  is not compact, since it has no largest element. However, it is limit point compact: Let  $A$  be an infinite subset of  $S_\Omega$ . Choose a subset  $B$  of  $A$  that is countably infinite. Being countable, the set  $B$  has an upper bound  $b$  in  $S_\Omega$ ; then  $B$  is a subset of the interval  $[a_0, b]$  of  $S_\Omega$ , where  $a_0$  is the smallest element of  $S_\Omega$ . Since  $S_\Omega$  has the least upper bound property, the interval  $[a_0, b]$  is compact. By the preceding theorem,  $B$  has a limit point  $x$  in  $[a_0, b]$ . The point  $x$  is also a limit point of  $A$ . Thus  $S_\Omega$  is limit point compact.

We now show these two versions of compactness coincide for metrizable spaces; for this purpose, we introduce yet another version of compactness called *sequential compactness*. This result will be used in Chapter 7.

**Definition.** Let  $X$  be a topological space. If  $(x_n)$  is a sequence of points of  $X$ , and if

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is an increasing sequence of positive integers, then the sequence  $(y_i)$  defined by setting  $y_i = x_{n_i}$  is called a *subsequence* of the sequence  $(x_n)$ . The space  $X$  is said to be *sequentially compact* if every sequence of points of  $X$  has a convergent subsequence.

**\*Theorem 28.2.** *Let  $X$  be a metrizable space. Then the following are equivalent:*

- (1)  $X$  is compact.
- (2)  $X$  is limit point compact.
- (3)  $X$  is sequentially compact.

*Proof.* We have already proved that (1)  $\Rightarrow$  (2). To show that (2)  $\Rightarrow$  (3), assume that  $X$  is limit point compact. Given a sequence  $(x_n)$  of points of  $X$ , consider the set  $A = \{x_n \mid n \in \mathbb{Z}_+\}$ . If the set  $A$  is finite, then there is a point  $x$  such that  $x = x_n$  for infinitely many values of  $n$ . In this case, the sequence  $(x_n)$  has a subsequence that is constant, and therefore converges trivially. On the other hand, if  $A$  is infinite, then  $A$  has a limit point  $x$ . We define a subsequence of  $(x_n)$  converging to  $x$  as follows: First choose  $n_1$  so that

$$x_{n_1} \in B(x, 1).$$

Then suppose that the positive integer  $n_{i-1}$  is given. Because the ball  $B(x, 1/i)$  intersects  $A$  in infinitely many points, we can choose an index  $n_i > n_{i-1}$  such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence  $x_{n_1}, x_{n_2}, \dots$  converges to  $x$ .

Finally, we show that (3)  $\Rightarrow$  (1). This is the hardest part of the proof.

First, we show that if  $X$  is sequentially compact, then the Lebesgue number lemma holds for  $X$ . (This would follow from compactness, but compactness is what we are trying to prove!) Let  $\mathcal{A}$  be an open covering of  $X$ . We assume that there is no  $\delta > 0$  such that each set of diameter less than  $\delta$  has an element of  $\mathcal{A}$  containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer  $n$ , there exists a set of diameter less than  $1/n$  that is not contained in any element of  $\mathcal{A}$ ; let  $C_n$  be such a set. Choose a point  $x_n \in C_n$ , for each  $n$ . By hypothesis, some subsequence  $(x_{n_i})$  of the sequence  $(x_n)$  converges, say to the point  $a$ . Now  $a$  belongs to some element  $A$  of the collection  $\mathcal{A}$ ; because  $A$  is open, we may choose an  $\epsilon > 0$  such that  $B(a, \epsilon) \subset A$ . If  $i$  is large enough that  $1/n_i < \epsilon/2$ , then the set  $C_{n_i}$  lies in the  $\epsilon/2$ -neighborhood of  $x_{n_i}$ ; if  $i$  is also chosen large enough that  $d(x_{n_i}, a) < \epsilon/2$ , then  $C_{n_i}$  lies in the  $\epsilon$ -neighborhood of  $a$ . But this means that  $C_{n_i} \subset A$ , contrary to hypothesis.

Second, we show that if  $X$  is sequentially compact, then given  $\epsilon > 0$ , there exists a finite covering of  $X$  by open  $\epsilon$ -balls. Once again, we proceed by contradiction. Assume that there exists an  $\epsilon > 0$  such that  $X$  cannot be covered by finitely many  $\epsilon$ -balls. Construct a sequence of points  $x_n$  of  $X$  as follows: First, choose  $x_1$  to be any point of  $X$ . Noting that the ball  $B(x_1, \epsilon)$  is not all of  $X$  (otherwise  $X$  could be covered by a single  $\epsilon$ -ball), choose  $x_2$  to be a point of  $X$  not in  $B(x_1, \epsilon)$ . In general, given  $x_1, \dots, x_n$ , choose  $x_{n+1}$  to be a point not in the union

$$B(x_1, \epsilon) \cup \dots \cup B(x_n, \epsilon),$$

using the fact that these balls do not cover  $X$ . Note that by construction  $d(x_{n+1}, x_i) \geq \epsilon$  for  $i = 1, \dots, n$ . Therefore, the sequence  $(x_n)$  can have no convergent subsequence; in fact, any ball of radius  $\epsilon/2$  can contain  $x_n$  for at most *one* value of  $n$ .

Finally, we show that if  $X$  is sequentially compact, then  $X$  is compact. Let  $\mathcal{A}$  be an open covering of  $X$ . Because  $X$  is sequentially compact, the open covering  $\mathcal{A}$  has a Lebesgue number  $\delta$ . Let  $\epsilon = \delta/3$ ; use sequential compactness of  $X$  to find a finite