MAT 320

4/23/19

updated [illegible]

#3

Last week: Power Series $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$, $\underbrace{a_n, z_0}_{\text{fixed}}, \underset{\text{variable}}{z} \in \mathbb{C}$

$\rightsquigarrow R = 1/\limsup |a_n|^{1/n}$

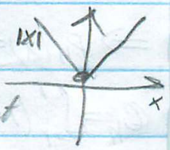
$f: B_R(z_0) \to \mathbb{C}$ well-defined and smooth

$\underbrace{f^{\langle k \rangle}}_{k\text{-th derivative}}(z) = \sum_{n=k}^{\infty} a_n n(n-1)\dots(n-k+1) z^{n-k}$

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $B_{R'}(z_0)$ $\forall R'<R$

$\Rightarrow$ good way to approximate $f(z)$ on $B_{R'}(z)$ with $R'>0$ by finite polynomials $\sum_{n=0}^{N} a_n(z-z_0)^N$,

Can we do this for $f(x)=|x|$ around x?



NO, b/c $|x|'$ does not exist at $x=0$, but $\sum_{n=0}^{\infty} a_n x^n$ is $\underbrace{\text{smooth}}$ on $B_R(0) \rightsquigarrow$ all derivatives exist

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#1 do not erase

Weierstrass Approximation Thm: For every $f:[a,b]\to\mathbb{C}$ cont., $\exists$ polynomials $p_n(x)=\sum_{k=1}^{n^*} a_{n;k}x^{k}$, $a_{n;k}\in\mathbb{C}$, s.t. $p_n \to f$ uniformly on $[a,b]$

If $f:[a,b]\to\mathbb{R}$, then $a_{n;k}$ can be taken in $\mathbb{R}$.

Lemma: $\forall n\in\mathbb{Z}^+$, $c_n \equiv \int_{-1}^{1}(1-x^2)^n dx \geq \frac{1}{\sqrt{n}}$

Sub-Lemma: $(1-x^2)^n \geq 1-nx^2 \quad \forall x\in[0,1]$

Pf: enough to show $(1-x)^n \geq 1-nx \quad \forall x\in[0,1]$

$LHS(0)=1=RHS(0)$

$LHS'(x) = -n(1-x)^{n-1} \geq -n \quad \underbrace{\forall x\in[0,1],\ n\geq 1}$

$RHS'(x) = -n \leq LHS'(x)$ — " —

$LHS(x) \geq RHS(0)$ — " —

Pf of Lemma: $c_n = 2\int_0^1 (1-x^2)^n dx \geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx$

$\overset{\text{Sublemma}}{\geq} 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx$

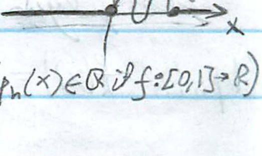
$= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$ ✓

do not erase #4

Pf of Thm: Assume $[a,b]=[0,1]$, $f(0), f(1)=0$

Extend to $\tilde{f}:\mathbb{R}\to\mathbb{C}$ by 0:

Take $p_n(x)=\frac{1}{c_n}\int_{-\infty}^{\infty}\tilde{f}(t+x)(1-t^2)^n dt$



(1) $\forall x\in\mathbb{R}$, $p_n(x)\in\mathbb{C}$ is well-defined ($p_n(x)\in\mathbb{R}$ if $f:[0,1]\to\mathbb{R}$)

(2) $p_n:\mathbb{R}\to\mathbb{C}$ is a polynomial

(3) $p_n \to f$ uniformly on $[0,1]$ $\Rightarrow$ done

#2

Pf of (1),(2) $\tilde{f}(t+x)=0$ unless $0\leq t+x\leq 1$

$\Rightarrow p_n(x)=\frac{1}{c_n}\underbrace{\int_{-x}^{1-x}\tilde{f}(t+x)(1-t^2)^n dt}_{\int \text{ of continuous function } g(t) \text{ on a bounded } [-x,1-x]} = \frac{1}{c_n}\underbrace{\int_0^1\tilde{f}(t)\left(1-(t-x)^2\right)^n dt}_{\sum_{k=0}^{2n} x^k \underbrace{\int_0^1 f_{n,k}(t)\,dt}_{\in\,\mathbb{C}}}$

$\Rightarrow p_n(x)\in\mathbb{C}$ is well-defined

Pn $p_n(x)$ is a polynomial in x

#3

Note: f cont. on $[0,1] \Rightarrow f$ uniformly cont. on $[0,1]$

$\Rightarrow \tilde{f}$ is uniformly continuous on $\mathbb{R}$

Pf of (3): let $\varepsilon>0$. Find $N>0$ s.t.

$|p_n(x)-f(x)| \leq 2\varepsilon \quad \forall n\geq N,\ x\in[0,1]$ $\quad (*)$

Take $M=\sup_{x\in\mathbb{R}}|\tilde{f}(x)| = \sup_{x\in[0,1]}|f(x)| \in \mathbb{R}$ b/c f cont. and $[0,1]$ cmpt

$\tilde{f}$ uniformly cont. $\Rightarrow \exists\,\delta>0$ s.t.

$|\tilde{f}(x)-\tilde{f}(x')|<\varepsilon$ if $|x-x'|<\delta$ $\quad (**)$

#4

(do not erase top 3 lines)

$x\in[0,1]$, $\tilde{f}(t+x)=0$ unless $0\le t+x\le 1$

$$\Rightarrow p_n(x)=\frac{1}{c_n}\int_{-1}^{1}\tilde{f}(t+x)(1-t^2)^n\,dt$$

$$c_n=\int_{-1}^{1}(1-t^2)^n\,dt \Rightarrow f(x)\equiv\tilde{f}(x)=\frac{1}{c_n}\int_{-1}^{1}\tilde{f}(x)(1-t^2)^n\,dt$$

#3

$$\therefore |p_n(x)-f(x)|\le \varepsilon+\frac{4M}{c_n}(1-\delta^2)^n\le$$

Lemma → $\le \varepsilon+4M\sqrt{n}(1-\delta^2)^n\le 2\varepsilon$ $(\cancel{***})$

$\forall n\ge$ some $N(M,\delta)$ b/c $\lim_{n\to\infty}\sqrt{n}(1-\delta^2)^n=0$

b/c $|1-\delta^2|<1$

$\therefore$ Given $\varepsilon>0$, choose $\delta=\delta(\varepsilon)>0$ from uniform cont. of f,

then choose $N=N(M,\delta(\varepsilon))$ s.t. $(***)$ $\forall x\in[0,1]$ ✓

#2

Proved case $\Rightarrow\exists$ polynomials $g_n:\mathbb{R}\to\mathbb{C}$ s.t.

$g_n\to g$ uniformly on $[0,1]$

Take $p_n(x)=g_n\left(\frac{x-a}{b-a}\right)+f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$

also polynomials

$$|p_n(x)-f(x)|=\left|g_n\left(\frac{x-a}{b-a}\right)-g\left(\underset{\in[0,1]}{\frac{x-a}{b-a}}\right)\right|$$

$g_n\to g$ uniformly on $[0,1]\Rightarrow p_n\to f$ uniformly on $[a,b]$

More general perspective: $X=$ set, $(Y,d_Y)=$ metric space

Maps$(X,Y)=$ maps from X to Y

$A\subset$ Maps(X,Y): some subcollections of maps from X to Y

The uniform closure of A is

$\overline{A}=\{f\in$ Maps(X,Y): $\exists$ sequence $f_n\in A$ s.t. $f_n\to f$ uniformly on $X\}\supset A$

HW10: $\overline{A}$ is "uniformly closed", i.e. $\overline{\overline{A}}=\overline{A}$

#2

$$c_n|p_n(x)-f(x)|=\left|\int_{-1}^{1}\left(\tilde{f}(t+x)-\tilde{f}(x)\right)(1-t^2)^n\,dt\right|$$

$$\le\int_{-1}^{1}|\tilde{f}(t+x)-\tilde{f}(x)|(1-t^2)^n\,dt$$

$$\le\int_{-\delta}^{\delta}\overset{(**)}{\varepsilon}(1-t^2)^n\,dt+\left\{\int_{-1}^{-\delta}+\int_{\delta}^{1}\right\}2\overset{\sup|\tilde{f}|}{M}\underbrace{(1-t^2)^n}_{\le(1-\delta^2)^n}\,dt$$

$$\le\varepsilon\cdot c_n+4M(1-\delta^2)^n$$

#4

(do not erase top line)

Have shown existence in this case.

Suppose $f:[a,b]\to\mathbb{C}$ any continuous map, $b>a$.

Take $g:[0,1]\to\mathbb{C}$, $g(x)=f(a+(b-a)x)-f(a)-(f(b)-f(a))x$

$\Rightarrow g$ cont., $g(0)=g(1)=0$

$$f(x)=g\left(\frac{x-a}{b-a}\right)+f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$

Ross: explicit formula for (different) p_n's

from [illegible] in $[a,b]=[0,1]$ case

from finitely many values $f\left(\frac{k}{n}\right)\in\mathbb{C}$, $k\in\mathbb{Z}$, $0\le k\le n$

good for applications/computing

but more technical pf.

Q: For what $A\subset$ Map(X,Y), is $\overline{A}=$ Maps(X,Y)?

More interesting: $(X,d_X)=$ metric space

$A\subset\mathcal{C}(X,Y)=\{$continuous functions $f:(X,d_X)\to(Y,d_Y)\}$

uniform convergence thm $\Rightarrow\overline{A}\subset\mathcal{C}(X,Y)$

Q: For what $A\subset\mathcal{C}(X,Y)$, is $\overline{A}=\mathcal{C}(X,Y)$

E.g. $A_{\mathbb{R}}\equiv\mathcal{P}([a,b],\mathbb{R})\subset\mathcal{C}([a,b],\mathbb{R})$ real polynomials

$A_{\mathbb{C}}\equiv\mathcal{P}([a,b],\mathbb{C})\subset\mathcal{C}([a,b],\mathbb{C})$ complex polynomials

WAT $\Rightarrow\overline{A_{\mathbb{R}/\mathbb{C}}}=\mathcal{C}([a,b],\mathbb{R}/\mathbb{C})$

MAT 320

$\Longrightarrow$ Color chalk $\Longleftarrow$

Typed: Crl of Weierstrass Approximation Thm: #4

$\forall a, \varepsilon \in \mathbb{R}^+$, $\exists$ real polyn. $p_1 : \mathbb{R} \to \mathbb{R}$ s.t.

$p(0)=0$, $|p(y) - |y|| < \varepsilon \ \forall y \in [-1,1]$ ⊛

Also: X = set, $A \subset Maps(X, \mathbb{R})$

$\rightsquigarrow \bar{A} \equiv \{f \in Maps(X,\mathbb{R}) : \exists f_n \in A$ s.t. $f_n \to f$ uniformly on $X\}$

$A \subset \bar{A}$, $\bar{\bar{A}} = \bar{A}$ (HW10)

Also need: $B(X,\mathbb{R}) \subset Maps(X,\mathbb{R})$ bounded functions

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Prp: Let $A \subset Maps(X,\mathbb{R})$ be an algebra. do not erase

(1) If A separates pts and does not vanish anywhere, #2

$\forall x_1, x_2 \in X, x_1 \neq x_2, c_1, c_2 \in \mathbb{R}$, $\exists f \in A$ s.t. $f(x_1) = c_1, f(x_2) = c_2$

(2) If $A \subset B(X,\mathbb{R}) = \{$bounded functions$\}$, then

$\bar{A} \subset B(X,\mathbb{R})$ is also an algebra

(3) If $\bar{A} \subset B(X,\mathbb{R})$, then $|f|, \max(f,g), \min(f,g) \in \bar{A} \ \forall f,g \in \bar{A}$

$\max(f,g) = \frac{1}{2}(f+g+|f-g|)$, $\min(f,g) = \frac{1}{2}(f+g-|f-g|)$ #4

$\bar{A}$ algebra, $|h| \in \bar{A} \ \forall h \in \bar{A} \Rightarrow \max(f,g), \min(f,g) \in \bar{A}$

$\forall f, g \in \bar{A}$

Pf of (2): $f_n \in B(X,\mathbb{R})$, $f_n \to f$ uniformly $\Longrightarrow f \to f_n$ uniformly

$\Longrightarrow$ if $A \subset B(X,\mathbb{R})$, then $\bar{A} \subset B(X,\mathbb{R})$

$f_n \to f$, $g_n \to g$ uniformly $\Rightarrow f_n + g_n \to f+g$, $f_n g_n \xrightarrow{②} fg$

$cf_n \to cf$ uniformly $\forall c \in \mathbb{R}$

Pf of (1): A separates $x_1, x_2 \Rightarrow \exists g \in A$ s.t. $g(x_1) \neq g(x_2)$ #4

A does not vanish at $x_1, x_2 \Rightarrow \exists f_1, f_2 \in A$ s.t. $f_1(x_1) \neq 0$, $f_2(x_2) \neq 0$

Let $h_1 = (g - g(x_2))f_1$, $h_2 = (g - g(x_1))f_2 : X \to \mathbb{R}$

$h_1(x_1), h_2(x_2) \neq 0$, $h_1(x_2), h_2(x_1) = 0$

Take $f = \frac{c_1}{h_1(x_1)} h_1 + \frac{c_2}{h_2(x_2)} h_2 : X \to \mathbb{R}$

$f(x_1) = c_1$, $f(x_2) = c_2$

$A \subset Maps(X,\mathbb{R})$ algebra, $g, f_1, f_2 \in A \Rightarrow h_1, h_2 \in A \Rightarrow f \in A$

4/25/18

#1 do not erase

Dfn: (1) $A \subset Maps(X,\mathbb{R})$ is an algebra over $\mathbb{R}$ if

$f+g$, $f \cdot g$, $cf \in A \ \forall f,g \in A, c \in \mathbb{R}$

(2) $A \subset Maps(X,\mathbb{R})$ separates points if

$\forall x_1, x_2 \in X; x_1 = x_2, \exists f \in A$ s.t. $f(x_1) \neq f(x_2)$

(3) $A \subset Maps(X,\mathbb{R})$ does not vanish anywhere if

$\forall x \in X$, $\exists f \in A$ s.t. $f(x) \neq 0$

E.g. $A = P([a,b],\mathbb{R}) = \{$polynomials$\}$ satisfy (1)-(3)

$|f|, \max(f,g), \min(f,g) : X \to \mathbb{R}$ #3

$|f|(x) = |f(x)|$, $\max(f,g)(x) = \max(f(x),g(x))$, $\min(f,g)(x) = \min(f(x),g(x))$

Pf of (3): $f \in \bar{A} \overset{(2)}{\subset} B(X,\mathbb{R}) \Longrightarrow a = \sup|f| \in \mathbb{R}^{\geq 0}$

Let $\varepsilon > 0 \overset{⊛}{\Longrightarrow} \exists c_1, \dots, c_k \in \mathbb{R}$ s.t. $|\sum_{i=1}^{k} c_i y^i - |y|| < \varepsilon \ \forall y \in [-a,a]$

$+ |f| \leq a \Rightarrow |\sum_{i=1}^{k} c_i f(x)^i - |f(x)|| < \varepsilon \ \forall x \in X$ ⊛⊛

$\bar{A}$ = algebra, $f \in \bar{A} \Rightarrow f_\varepsilon \equiv \sum_{i=1}^{k} c_i f^i \in \bar{A} \ \forall \varepsilon > 0$

⊛⊛ $\Longrightarrow f_{1/n} \to f$ uniformly; $\bar{\bar{A}} = \bar{A} \Longrightarrow f \in \bar{A}$ ✓

$\Longrightarrow$ if $A \subset B(X,\mathbb{R})$ is algebra, then $\bar{A} \subset B(X,\mathbb{R})$ is algebra #3

only ② needs: $f_n, g \in B(X,\mathbb{R})$

$|f_n(x) g_n(x) - f(x) g(x)| \leq |f_n(x)(g_n(x) - g(x))| + |(f_n(x) - f(x)) g(x)|$

$$\leq (\sup|f_n|) \underbrace{|g_n(x) - g(x)|}_{<\varepsilon} + (\sup|g|) \underbrace{|f_n(x) - f(x)|}_{<\varepsilon}$$

$\forall n \geq$ some $N(\varepsilon)$, $x \in X$

doesn't depend on x

Stone-Weierstrass Approximation Thm #3

Let (X, d_X) be a cmpt metric space, $C(X,\mathbb{R}) = \{$continuous $f : X \to \mathbb{R}\}$

If $A \subset C(X,\mathbb{R})$ is an algebra that separates pts in X and does not vanish anywhere on X, then $\bar{A} = C(X,\mathbb{R})$.

Pf: uniform convergence thm $\Longrightarrow \bar{A} \subset C(X,\mathbb{R})$;

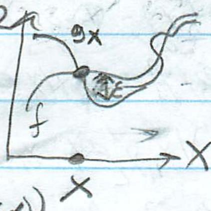
Let $f \in C(X,\mathbb{R})$, $\varepsilon > 0$. Find $f_\varepsilon \in \bar{A}$ s.t.

$|f_\varepsilon(x) - f(x)| < \varepsilon \ \forall x \in X$) $\Longrightarrow f \in \bar{\bar{A}}$

#4

Claim: $\forall x \in X, \exists g_x \in \bar{A}$ s.t. $g_x(x) = f(x)$, $g_x(x') > f(x') - \varepsilon \ \forall x' \in X$

Assume claim.



$\forall x \in X$, $\mathcal{U}_x \equiv \{x' \in X : g_x(x') < f(x') + \varepsilon\}$

is open in X (preimage of $(-\infty, \varepsilon)$ by

cont $X \to \mathbb{R}$, $x' \to g_x(x') - f(x')$)

$x \in \mathcal{U}_x$ b/c $g_x(x) = f(x) \Rightarrow \{\mathcal{U}_x : x \in X\}$ is an open cover

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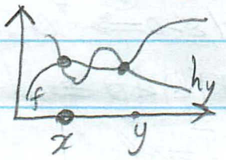
#4

(do not erase top line)

Pf of Claim: A algebra that separates pts and does not vanish on X

$\xrightarrow{Prp(2)}$ so is $\bar{A} \supset A \xrightarrow{Prp(1)} \forall y \in X, \exists h_y \in \bar{A}$ s.t. $h_y(x) = f(x)$, $h_y(y) = f(y)$

$\forall y \in X$, $\mathcal{U}_y \equiv \{x' \in X : h_y(x') > f(x') - \varepsilon\}$



is open in X (preimage of $(-\varepsilon, \infty)$ by cont.

cont. $h_y - f : X \to \mathbb{R}$

$y \in \mathcal{U}_y$ b/c $h_y(y) = f(y) \Rightarrow \{\mathcal{U}_y : y \in X\}$ is an open cover of X

#4

$\mathbb{C}$-case: $A \subset \mathrm{Maps}(X, \mathbb{C})$ is an algebra over $\mathbb{C}$

if $f+g$, $f \cdot g$, $cf \in A$ $\forall f, g \in A$, $c \in \mathbb{C}$

Such algebra is self-adjoint if $\bar{f} \in A \quad \forall f \in A$

$\bar{f} : X \to \mathbb{C}$, $\bar{f}(x) \equiv \overline{f(x)}$

$f : X \to \mathbb{C} \rightarrow f = u + iv$, $u, v : X \to \mathbb{R}$

$u = \frac{1}{2}(f + \bar{f})$, $v = \frac{1}{2i}(f - \bar{f})$

$\Rightarrow$ if $f \in A$ and A is self-adjoint algebra over $\mathbb{C}$, then $u, v \in A$

#2

Thm: Let (X, d_X) be a cmpt metric space, $\mathcal{C}(X, \mathbb{C}) = \{f : X \to \mathbb{C}_{\text{cont.}}\}$

If $A \subset \mathcal{C}(X, \mathbb{C})$ is a self-adjoint algebra over $\mathbb{C}$ that separates pts and does not vanish on $\mathbb{C}$, then $\bar{A} = \mathcal{C}(X, \mathbb{C})$

Pf: last board $\Rightarrow A_{\mathbb{R}} = A \cap \mathrm{Maps}(X, \mathbb{R}) \subset \mathcal{C}(X, \mathbb{R})$

an algebra over $\mathbb{R}$ that separates pts and does not vanish on X

$\Rightarrow \overline{A_{\mathbb{R}}} = \mathcal{C}(X, \mathbb{R})$, $\overline{A_{\mathbb{R}}} = \bar{A} \cap \mathrm{Maps}(X, \mathbb{R}) = \bar{A}_{\mathbb{R}}$

$\bar{A}$ $\mathbb{C}$-algebra, $\bar{A}_{\mathbb{R}} = \mathcal{C}(X, \mathbb{R}) \Rightarrow \bar{A} = \mathcal{C}(X, \mathbb{C})$

#1

X cmpt $\Rightarrow X \subset \mathcal{U}_{x_1} \cup \ldots \cup \mathcal{U}_{x_m}$ for some $x_1, \ldots, x_m \in X$

Take $f_\varepsilon = \min(g_{x_1}, \ldots, g_{x_m}) \in \bar{A}$ by Prp (3)

$\Rightarrow \forall x' \in \mathcal{U}_{x_i}, i = 1, \ldots, n : \underbrace{f(x') - \varepsilon}_{\text{true } \forall g_{x_j} \text{ by choice in Claim}} < f_\varepsilon(x') \le g_{x_i}(x') < f(x') + \varepsilon$ ← by choice of $\mathcal{U}_{x_i}$

$X \subset \mathcal{U}_{x_1} \cup \ldots \cup \mathcal{U}_{x_m} \Rightarrow f(x') - \varepsilon < f_\varepsilon(x') < f(x) + \varepsilon$ ✓

#1

X cmpt $\Rightarrow X \subset \mathcal{U}_{y_1} \cup \ldots \cup \mathcal{U}_{y_m}$ for some $y_1, \ldots, y_m \in X$

Take $g_x = \max(h_{y_1}, \ldots, h_{y_m}) \in \bar{A}$ by Prp (3)

$h_{y_i}(x) = f(x) \ \forall i \Rightarrow g_x(x) = f(x)$

$\forall x' \in \mathcal{U}_{y_i}, i = 1, \ldots, n : g_x(x') \ge h_{y_i}(x') > f(x') - \varepsilon$ ← choice of $\mathcal{U}_{x_i}$

$X \subset \mathcal{U}_{x_1} \cup \ldots \cup \mathcal{U}_{x_m} \Rightarrow g_x(x') > f(x') - \varepsilon$

$\Longrightarrow$ Claim ✓

#1

Prp(1) applies to algebra A over $\mathbb{C}$ with $c_1, c_2 \in \mathbb{C}$

that separates pts and does not vanish on X

$\Rightarrow \forall x_1, x_2 \in X$, $\exists f \in A \subset \mathrm{Maps}(X, \mathbb{C})$ s.t. $f(x_1) = 1$, $f(x_2) = 0$

if A is self-adjoint, then $u = \mathrm{Re} f \in A$, so $u(x_1) = 1$, $u(x_2) = 0$

$\Rightarrow A_{\mathbb{R}} \equiv A \cap \mathrm{Map}(X, \mathbb{R})$ separates points and does not vanish on X.

A $\mathbb{C}$-algebra $\Rightarrow A_{\mathbb{R}}$ is $\mathbb{R}$-algebra

---

b/c $f = u + iv$ ← anything in $\bar{A}$, $\mathcal{C}(X, \mathbb{C})$

u, v: anything in $\bar{A}_{\mathbb{R}} = \mathcal{C}(X, \mathbb{R})$

$f = u + iv : X \to \mathbb{C}$ cont iff $u, v : X \to \mathbb{R}$ cont

Rudin's Principles of Mathematical Analysis

THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n \to \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \leq x \leq 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

$$Q_n(x) = c_n(1 - x^2)^n \qquad (n = 1, 2, 3, \ldots), \tag{47}$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)\, dx = 1 \qquad (n = 1, 2, 3, \ldots). \tag{48}$$

We need some information about the order of magnitude of c_n. Since

$$\begin{aligned} \int_{-1}^{1} (1 - x^2)^n\, dx = 2\int_0^1 (1 - x^2)^n\, dx \geq 2\int_0^{1/\sqrt{n}} (1 - x^2)^n\, dx \\ \geq 2\int_0^{1/\sqrt{n}} (1 - nx^2)\, dx \\ = \frac{4}{3\sqrt{n}} \\ > \frac{1}{\sqrt{n}}, \end{aligned}$$

it follows from (48) that

$$c_n < \sqrt{n}. \tag{49}$$

The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

$$Q_n(x) \leq \sqrt{n}\,(1 - \delta^2)^n \qquad (\delta \leq |x| \leq 1), \tag{50}$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

$$P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t)\,dt \qquad (0 \leq x \leq 1). \tag{51}$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t)\,dt = \int_0^1 f(t)Q_n(t - x)\,dt,$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x + t) - f(x)]Q_n(t)\,dt \right| \\
&\leq \int_{-1}^{1} |f(x + t) - f(x)|\,Q_n(t)\,dt \\
&\leq 2M\int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt + 2M\int_{\delta}^{1} Q_n(t)\,dt \\
&\leq 4M\sqrt{n}\,(1 - \delta^2)^n + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

for all large enough n, which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

7.27 Corollary *For every interval* $[-a, a]$ *there is a sequence of real polynomials* P_n *such that* $P_n(0) = 0$ *and such that*

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Proof By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \qquad (n = 1, 2, 3, \ldots)$$

have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

7.28 Definition A family $\mathscr{A}$ of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathscr{A}$, (ii) $fg \in \mathscr{A}$, and (iii) $cf \in \mathscr{A}$ for all $f \in \mathscr{A}$, $g \in \mathscr{A}$ and for all complex constants c, that is, if $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c.

If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be *uniformly closed.*

Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the *uniform closure* of $\mathscr{A}$. (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

7.29 Theorem *Let* $\mathscr{B}$ *be the uniform closure of an algebra* $\mathscr{A}$ *of bounded functions. Then* $\mathscr{B}$ *is a uniformly closed algebra.*

Proof If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathscr{A}$, $g_n \in \mathscr{A}$. Since we are dealing with bounded functions, it is easy to show that

$$f_n + g_n \to f + g, \qquad f_n g_n \to fg, \qquad cf_n \to cf,$$

where c is any constant, the convergence being uniform in each case.

Hence $f + g \in \mathscr{B}$, $fg \in \mathscr{B}$, and $cf \in \mathscr{B}$, so that $\mathscr{B}$ is an algebra.

By Theorem 2.27, $\mathscr{B}$ is (uniformly) closed.

7.30 Definition Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ *vanishes at no point of* E.

The algebra of all polynomials in one variable clearly has these properties on R^1. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f.

The following theorem will illustrate these concepts further.

7.31 Theorem *Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that*

$$f(x_1) = c_1, \qquad f(x_2) = c_2.$$

Proof The assumptions show that $\mathscr{A}$ contains functions g, h, and k such that

$$g(x_1) \neq g(x_2), \qquad h(x_1) \neq 0, \qquad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \qquad v = gh - g(x_2)h.$$

Then $u \in \mathscr{A}$, $v \in \mathscr{A}$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

We shall divide the proof into four steps.

STEP 1 *If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.*

Proof Let

$$a = \sup |f(x)| \qquad (x \in K) \tag{52}$$

and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers $c_1, \ldots, c_n$ such that

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \qquad (-a \le y \le a). \tag{53}$$

Since $\mathscr{B}$ is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of $\mathscr{B}$. By (52) and (53), we have

$$\left| g(x) - |f(x)| \right| < \varepsilon \qquad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this shows that $|f| \in \mathscr{B}$.

STEP 2 *If* $f \in \mathscr{B}$ and $g \in \mathscr{B}$, *then* $\max(f, g) \in \mathscr{B}$ *and* $\min(f, g) \in \mathscr{B}$.

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \ge g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min(f, g)$ is defined likewise.

Proof Step 2 follows from step 1 and the identities

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \ldots, f_n \in \mathscr{B}$, then $\max(f_1, \ldots, f_n) \in \mathscr{B}$, and

$$\min(f_1, \ldots, f_n) \in \mathscr{B}.$$

STEP 3 *Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and*

$$g_x(t) > f(t) - \varepsilon \qquad (t \in K). \tag{54}$$

Proof Since $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}$ satisfies the hypotheses of Theorem 7.31 so does $\mathscr{B}$. Hence, for every $y \in K$, we can find a function $h_y \in \mathscr{B}$ such that

$$h_y(x) = f(x), \qquad h_y(y) = f(y). \tag{55}$$

By the continuity of h_y there exists an open set J_y, containing y, such that

$$h_y(t) > f(t) - \varepsilon \qquad (t \in J_y). \tag{56}$$

Since K is compact, there is a finite set of points $y_1, \ldots, y_n$ such that

$$K \subset J_{y_1} \cup \cdots \cup J_{y_n}. \tag{57}$$

Put

$$g_x = \max(h_{y_1}, \ldots, h_{y_n}).$$

By step 2, $g_x \in \mathscr{B}$, and the relations (55) to (57) show that g_x has the other required properties.

STEP 4 *Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathscr{B}$ such that*

$$|h(x) - f(x)| < \varepsilon \qquad (x \in K). \tag{58}$$

Since $\mathscr{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof Let us consider the functions g_x, for each $x \in K$, constructed in step 3. By the continuity of g_x, there exist open sets V_x containing x, such that

$$g_x(t) < f(t) + \varepsilon \qquad (t \in V_x). \tag{59}$$

Since K is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that

$$K \subset V_{x_1} \cup \cdots \cup V_{x_m}. \tag{60}$$

Put

$$h = \min(g_{x_1}, \ldots, g_{x_m}).$$

By step 2, $h \in \mathscr{B}$, and (54) implies

$$h(t) > f(t) - \varepsilon \qquad (t \in K), \tag{61}$$

whereas (59) and (60) imply

$$h(t) < f(t) + \varepsilon \qquad (t \in K). \tag{62}$$

Finally, (58) follows from (61) and (62).

Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on $\mathscr{A}$, namely, that $\mathscr{A}$ be *self-adjoint*. This means that for every $f \in \mathscr{A}$ its complex conjugate $\bar{f}$ must also belong to $\mathscr{A}$; $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

7.33 Theorem *Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense $\mathscr{C}(K)$.*

> **Proof** Let $\mathscr{A}_R$ be the set of all real functions on K which belong to $\mathscr{A}$.
>
> If $f \in \mathscr{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since $\mathscr{A}$ is self-adjoint, we see that $u \in \mathscr{A}_R$. If $x_1 \neq x_2$, there exists $f \in \mathscr{A}$ such that $f(x_1) = 1, f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that $\mathscr{A}_R$ separates points on K. If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathscr{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$; hence $\mathscr{A}_R$ vanishes at no point of K.
>
> Thus $\mathscr{A}_R$ satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on K lies in the uniform closure of $\mathscr{A}_R$, hence lies in $\mathscr{B}$. If f is a complex continuous function on K, $f = u + iv$, then $u \in \mathscr{B}$, $v \in \mathscr{B}$, hence $f \in \mathscr{B}$. This completes the proof.

EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.
3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?