MAT 320: Introduction to Analysis, Spring 2019
Baire Spaces

Let (X,d) be a metric space and $A \subset X$. The interior of A, or $\operatorname{Int} A$, is the largest open subset of X contained in A; this is the union of all open subsets contained in A. The interior of A is empty if and only if no nonempty open subset U of X is contained in A, i.e. every nonempty open subset U of X intersects $X - A$. The last condition means that $X - A$ is dense in X.

A metric space (X,d) is called Baire if the intersection

$$\bigcap_{n=1}^{\infty} U_n \subset X$$

is dense in (X,d) for every sequence $U_1, U_2, \ldots \subset X$ of dense open subsets of (X,d); this is Ross's property 21.7a. This is equivalent to the condition that the union

$$\bigcup_{n=1}^{\infty} F_n \subset X$$

of closed sets $F_1, F_2, \ldots \subset X$ with empty interiors has empty interior; this is Ross's property 21.7b. This condition is in turn equivalent to the condition that no nonempty open subset $W \subset X$ is a countable union of nowhere dense subsets of X, i.e. every open subset $W \subset X$ is of Category 2.

The equivalence of the first two conditions above is obtained as follows. Let $U_1, U_2, \ldots \subset X$ be any sequence of subsets and $B_n = X - U_n$. Each set U_n is open (resp. dense) in X if and only if each set B_n is closed (resp. has empty interior in X); the second equivalence is by the first paragraph above. The intersection of the sets U_n is dense in X if and only if its complement

$$X - \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (X - U_n) = \bigcup_{n=1}^{\infty} B_n$$

has empty interior.

Baire Category Theorem. A complete metric space is a Baire space.

This implies that a complete metric space (X,d) is of Category 2 in itself. This is the statement of Ross's Theorem 21.8, i.e. this theorem is a corollary of the usual formulation of Baire Category Theorem, which is much weaker than the theorem itself. For example, let (X,d) be a metric space consisting of $\mathbb{Q}$ and another isolated point p^*, e.g.

$$X = \mathbb{Q} \sqcup \{p^*\}, \qquad d(x,x') = \begin{cases} |x - x'|, & \text{if } x, x' \in \mathbb{Q}; \\ 1, & \text{if } x \neq x',\ p^* \in \{x, x'\}; \\ 0, & \text{if } x = x'. \end{cases}$$

A nowhere dense subset F in this space cannot contain p^* (because $\{p^*\}$ is open in this metric space). Thus, (X,d) is of Category 2. However, the open subset $\mathbb{Q}$ of X is not of the second category, since it is a countable union of its own points, which are nowhere dense in $\mathbb{Q}$ and X.