MAT 312/AMS 351: Applied Algebra
Solutions to Problem Set 7 (15pts)

5.3 2; 3pts *Let $H \subset S_4$ be the subgroup consisting of the permutations* id, (12)(34), (13)(24), (14)(23) *and G be the group of rigid symmetries of a rectangle, which is not a square. Describe all isomorphisms between H and G.*

Label the vertices of the rectangle by 1,2,3,4 in a circular order. The group G consists of the identity id, the reflection σ about the line joining the centers of the edges (12) and (34), the reflection τ about the line joining the centers of the edges (14) and (23), and the rotation $R = \sigma\tau$ by π about the center which interchanges the diagonally opposite vertices. An isomorphism $f : H \longrightarrow G$ is determined by where f sends (12)(34) and (14)(23) because

$$f(\mathrm{id}) = \mathrm{id} \qquad \text{and} \qquad f\big((13)(24)\big) = f\big((12)(34)\circ(14)(23)\big) = f\big((12)(34)\big)\circ f\big((14)(23)\big).$$

The elements $f((12)(34))$ and $f((14)(23))$ of G must be distinct from id and from each other. This gives us 3 choices for $f((12)(34))$, i.e. σ, τ, R, and the 2 remaining choices for $f((14)(23))$. In particular, there are 6 isomorphisms between H and G.

5.3 10; 6pts *Let G be a non-abelian group of order 8. Show that G is isomorphic to either the dihedral group D_4 or the quaternion group $\mathbb{H}_0$.*

Since the order $\mathfrak{o}(a)$ of every element a of G divides $|G| = 8$, the only possibilities for $\mathfrak{o}(a)$ are 1,2,4,8. If G contains an element a of order 8, then G is a cyclic group of order 8 generated by a and is thus abelian. If G contains no element of order 4 or 8, then $a^2 = e$ for all $e \in G$, which again implies that G is abelian (see 4.3 3 on HW5). Since G is assumed to be non-abelian, G thus contains an element a of order 4 and no element of order 8.

Let $a \in G$ be an element of order 4. Thus, e, a, a^2, a^3 are four distinct elements of G and $a^4 = e$. Let $b \in G$ be a fifth element. Then,

$$\begin{aligned}
ab &\neq e, a, a^2, a^3, b && \text{b/c} \quad b \neq a^3, e, a, a^2, \quad a \neq e;\\
a^2b &\neq e, a, a^2, a^3, b, ab && \text{b/c} \quad b \neq a^2, a^3, e, a, \quad a^2, a \neq e;\\
a^3b &\neq e, a, a^2, a^3, b, ab, a^2b && \text{b/c} \quad b \neq a, a^2, a^3, e, \quad a^3, a^2, a \neq e.
\end{aligned}$$

Thus, $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ with $a^4 = e$. It remains to determine what b^2 and ba are.

Since $b \neq e, a, a^2, a^3$ and $b \neq a^3, e, a, a^2$,

$$b^2 \neq b, ab, a^2b, a^3b \qquad \text{and} \qquad ba \neq e, a, a^2, a^3, b,$$

respectively. If $b^2 = a, a^3$, then G is cyclic generated by b and thus abelian, contrary to the assumption. Thus, either $b^2 = e$ or $b^2 = a^2$. Furthermore, $ba \neq ab, a^2b$; otherwise, $(ba)^2$ would equal either a or a^3 and generate G, contrary to the assumption that G is not abelian. Thus, $ba = a^3b$ in both cases. If $b^2 = e$, we thus get the dihedral group D_4. If $b^2 = a^2$, we get the quaternion group $\mathbb{H}_0$ with $i = a$, $j = b$, and $k = ab$.

Alternative solution. Let $a\in G$ be an element of order 4. Thus, $a^4=e$ and $\langle a\rangle$ is a subgroup of G of order 4. Let $b\in G$ be an element not in $\langle a\rangle$. Thus, the right cosets $\langle a\rangle e=\langle a\rangle$ and $\langle a\rangle b$ are distinct and thus disjoint. Since each of them contains 4 elements and $|G|=8$,

$$G=\langle a\rangle\sqcup\langle a\rangle b=\{e,a,a^2,a^3,b,ab,a^2b,a^3b\}.$$

Since $\langle a\rangle\neq\langle a\rangle b$, $\langle a\rangle b\neq\langle a\rangle b^2$ and $b^2\notin\langle a\rangle b$. Since $b\notin\langle a\rangle$ and $\langle a\rangle$ is closed under inverses and multiplication, $ba\notin\langle a\rangle$. Since $b\neq e$, $b^2\neq b$. Now continue with *If* $b^2=a,a^3$*, then* in the first solution.

5.3 5; 2pts *Let G and H be two groups. Show that $G\times H$ is abelian if and only if G and H are abelian.*

Suppose the groups G and H are abelian. If $g,g'\in G$ and $h,h'\in H$, then

$$(g,h)(g',h')=(gg',hh')=(g'g,h'h)=(g',h')(g,h).$$

The first and last equalities above hold by the definition of the product in the group $G\times H$; the middle equality holds by the assumption that G and H are abelian. Since the first and fourth expressions above are the same for all $(g,h),(g',h')\in G\times H$, i.e. all possible elements of $G\times H$, the group $G\times H$ is abelian.

Suppose the group $G\times H$ is abelian. If $g,g'\in G$ and $h,h'\in H$, then

$$(gg',hh')=(g,h)(g',h')=(g',h')(g,h)=(g'g,h'h).$$

The first and last equalities above hold by the definition of the product in the group $G\times H$; the middle equality holds by the assumption that $G\times H$ is abelian. The equality of the first and last expressions above means that $gg'=g'g$ and $hh'=h'h$. Since these equalities holds for all $g,g'\in G$ and $h,h'\in H$ *and* both sets are nonempty (because groups are never empty), it follows that the groups G and H are abelian.

5.3 9; 4pts *Show that the group $A_4\subset S_4$ of even permutations of 4 elements contains no subgroup of order 6.*

Every permutation $\pi\in S_4$ is a product of disjoint cycles. Thus, π can be only the identity id ("trivial" product), a transposition, product of two disjoint transpositions, cycle of order 3, or cycle of order 4. Since transpositions and cycles of order 4 are odd permutations, A_4 consists of id, products of permutations $(ab)(cd)$, and order 3 cycles (abc) for *distinct* elements a,b,c,d of $\{1,2,3,4\}$. The only elements of A_4 of order 2 are the products of transpositions $(ab)(cd)$; there are 3 such elements (since disjoint permutations commute, $(ab)(cd)=(cd)(ab)$). A product of two distinct such elements is

$$(ab)(cd)\circ(ac)(bd)=(ad)(bc)$$

is the third elements of order 2. Thus, the product of any two distinct elements of order 2 of A_4 is an element of order 2.

Every group of order 6 is isomorphic to either the cyclic group C_6 or the symmetric group S_3. A subgroup $H\subset A_4$ of order 6 would have to be isomorphic to one of them. The group S_3 consists of the identity, the three transpositions (xy), and the two order 3 cycles (xyz). Thus, S_3 contains precisely three order 2 elements (the transpositions). The product of any two such elements is an order 3 cycle. Thus, H cannot be isomorphic to S_3 by the end of the previous paragraph. The group C_6 contains an element of order 6. Since A_4 contains no element of order 6, H cannot be isomorphic to C_6 either. It follows that A_4 contains no subgroup H of order 6.