MAT 312/AMS 351: Applied Algebra
Solutions to Problem Set 6 (13pts)

5.1 1; 2pts *Let G be a group and $a,b \in G$. Show that $ab=ba$ if and only if $(ab)^{-1}=a^{-1}b^{-1}$.*

For all $a,b \in G$, $(ab)^{-1} = b^{-1}a^{-1}$ and $(ba)^{-1} = a^{-1}b^{-1}$. By the uniqueness of the inverses, this implies that $ab=ba$ if and only if $b^{-1}a^{-1}=a^{-1}b^{-1}$.

5.1 9; 3pts *Let G be a group. Define a relation on G by $a \sim b$ if there exists $g \in G$ such that $b=g^{-1}ag$. Show that $\sim$ is an equivalence relation.*

For every $a \in G$, $a=e^{-1}ae$ and so $a\sim a$; thus, $\sim$ is reflexive. For all $a,b \in G$ such that $a\sim b$, there exists $g \in G$ such that $b=g^{-1}ag$. Thus, $a=(g^{-1})^{-1}bg^{-1}$ and so $\sim$ is symmetric. For all $a,b,c \in G$ such that $a\sim b$ and $b\sim c$, there exist $g,h \in G$ such that $b=g^{-1}ag$ and $c=h^{-1}bh$. Thus,

$$c = h^{-1}bh = h^{-1}(g^{-1}ag)h = (h^{-1}g^{-1})a(gh) = (gh)^{-1}a(gh)$$

and so $\sim$ is transitive.

5.2 2; 2pts *Show that $|G_n|$ is even for every $n \geq 3$.*

Since $[-1]_n \neq [1]_n$ in $G_n \subset \mathbb{Z}_n$ for $n\geq 3$ and $(-1)^2=1$, the order of the element $[-1]_n$ of $(G_n,\cdot)$ is 2. Since the order of an element of a finite group divides the order of the group, 2 divides $|G_n|$ for $n\geq 3$.

5.2 4; 3pts *Let H be a subgroup of a group G, $a \in G$, and $b \in aH$. Show that*

$$H = \{b^{-1}c \colon c \in aH\}.$$

Since $b \in H$, $b=ah$ for some $h \in H$. If $c \in aH$, $c=ah'$ for some $h' \in H$. Thus,

$$b^{-1}c = (ah)^{-1}(ah') = h^{-1}a^{-1}ah' = h^{-1}h' \in H,$$

since $H \subset G$ is closed under multiplication. This implies that

$$H \supset \{b^{-1}c \colon c \in aH\}.$$

If $h' \in H$, then $hh' \in H$ and $ahh' \in aH$. Since

$$b^{-1}(ahh') = (ah)^{-1}(ahh') = h^{-1}a^{-1}ahh' = h^{-1}hh' = h',$$

it follows that

$$H \subset \{b^{-1}c \colon c \in aH\}.$$

Problem C (3pts)

Determine the elements of the cyclic subgroup of $\mathrm{GL}_n\mathbb{Z}$ *generated by the matrix*

$$g \equiv \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

explicitly.

The elements are $\mathbb{I}_2, g, g^2, g^3, g^4, g^5$, where

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad g^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$
$$g^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g^4 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad g^5 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$