MAT 312/AMS 351: Applied Algebra
Solutions to Problem Set 4 (15pts)

4.2 4; 2pts *Let π, σ be permutations of a set X such that $(\pi\sigma)^2 = \pi^2\sigma^2$. Show that $\pi\sigma = \sigma\pi$.*

By assumption, $\pi\sigma\pi\sigma = \pi\pi\sigma\sigma$. Multiplying both sides by π^{-1} on LHS and σ^{-1} on RHS, we obtain

$$\pi^{-1}\pi\sigma\pi\sigma\sigma^{-1} = \pi^{-1}\pi\pi\sigma\sigma\sigma^{-1}.$$

Since $\pi^{-1}\pi, \sigma\sigma^{-1} = \mathrm{id}_X$, the above reduces to $\sigma\pi = \pi\sigma$. Note that this argument uses the associativity of the multiplication (composition) of transpositions multiple times.

4.2 8; 6pts *Let π be a permutation of a set X and $r, s \in \mathbb{Z}^+$. Show that*

$$\text{(ii) } (\pi^r)^s = \pi^{rs} \qquad \text{(iii) } \pi^{-r} = (\pi^r)^{-1}.$$

(ii) We prove this by induction on $s \in \mathbb{Z}^+$. The base case, $s = 1$, is just $(\pi^r)^1 = \pi^r$ and is true by the definition of LHS. For the inductive step, assume that (ii) holds for some $s \in \mathbb{Z}^+$. Then,

$$(\pi^r)^{s+1} = (\pi^r)(\pi^r)^s = \pi^r\pi^{rs} = \pi^{r+rs} = \pi^{r(s+1)};$$

the first equality above holds by definition, the second by the inductive assumption, and the third by (i). By the equality of LHS and RHS above, (ii) holds for $s+1$. This completes the proof.

(iii) We prove this by induction on $r \in \mathbb{Z}^+$. The base case, $r = 1$, is just $\pi^{-1} = (\pi^1)^{-1}$ and is true by the definition of RHS. For the inductive step, assume that (ii) holds for some $r \in \mathbb{Z}^+$. Then,

$$\pi^{-(r+1)} = (\pi^{-1})^{r+1} = (\pi^{-1})^1(\pi^{-1})^r = \pi^{-1}(\pi^r)^{-1} = (\pi^r\pi)^{-1} = (\pi^{r+1})^{-1};$$

the first two equalities above hold by definition, the third by the definition and the inductive assumption, the fourth by standard set theory, and the fifth by (i). By the equality of LHS and RHS above, (iii) holds for $r+1$. This completes the proof.

4.2 10; 3pts (answer only) *Show that every element of S_n is a product of transpositions of the form $(k\ k+1)$*

By Theorem 4.2.11, every element π of S_n is a product of transpositions $(k\ k+r)$. Thus, it is sufficient to show that every transpositions $(k\ k+r)$ with $k, r \geq 1$ is a product of "basic" transpositions $(m\ m+1)$. We prove this by induction on r. In the base $r = 1$ case, $(k\ k+1)$ is already a basic transposition and there is nothing to prove. Suppose $(k\ k+r)$ is a product of basic transpositions for some $r \geq 1$. Since

$$\big(k\ k+r+1\big) = \big(k+r\ k+r+\big)\big(k\ k+r\big)\big(k+r\ k+r+1\big),$$

it follows that $(k\ k+r+1)$ is also a product of basic transpositions. Thus, every transposition $(k\ k+r)$ is a product of basic transpositions.

4.2 13; 4pts *Show that the order of pieces in a 4×4 puzzle with one blank as in Figure 4.5 can never be reversed.*

For a permutation π of the 15 pieces and the blank, let $r(\pi)$ and $c(\pi)$ be the row and column of the position of the blank piece. Define

$$\mathfrak{s}(\pi) = \begin{cases} 0, & \text{if } r(\pi)+c(\pi) \text{ is even;} \\ 1, & \text{if } r(\pi)+c(\pi) \text{ is odd.} \end{cases}$$

The value of $\mathfrak{s}(\pi)$ flips with every move in the puzzle. Since the blank starts at the bottom right corner, it can thus end at the same place only after an even number of moves (because the value of $\mathfrak{s}$ returns to the original one). Since every move in the puzzle is a transposition, it follows that every achievable permutation of the pieces in the puzzle that returns the blank to the bottom right corner is a product of an even number of transpositions and thus has sign 1.

On the other hand, the permutation

$$1, 2, \ldots, 14, 15, \text{blank} \qquad \longrightarrow \qquad 15, 14, \ldots, 2, 1, \text{blank}$$

from the left diagram of Figure 4.5 to the right can be written as a product of

$$14 + 13 + \ldots + 2 + 1 = \binom{15}{2} = 105$$

transpositions (it takes 14 switches to move 15 to the leftmost spot, 13 switches to move 14 to the second spot from the left, etc.). Since this permutation returns the blank to the bottom right corner and has sign -1, it is not an achievable permutation of the pieces in the puzzle.