MAT 127 LECTURE OUTLINE WEEK 9

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: This week, we will introduce and study more tests for convergence: the comparison test, the limit comparison test, the ratio test, and the root test. Actually, the last three of these tests are just applications of the comparison test, but they are often easier to apply in practice. [The textbook also has a section on the alternating series test, but this will be covered next week and will not appear on Midterm 2.]

(1) The next main convergence test is called the **comparison test**. The idea behind the comparison test is simple: If a given series converges, then any series that is smaller also converges. If a given series diverges, then any series that is bigger also diverges. The exact statement is Theorem 5.11 in the book.

> Comparison test.
> (a) Suppose that $0 \leq a_n \leq b_n$ for all $n \geq N$ for some integer N.
> If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
> (b) Suppose that $a_n \geq b_n \geq 0$ for all $n \geq N$ for some integer N.
> If $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

(2) A common situation is where a_n is a rational function. In this case, we are able to compare $\sum_{n=1}^{\infty} a_n$ with a p-series (or multiple of a series). Here's an example.

Example. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

The idea is to observe that $\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$ for all n, recalling that the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges. One thing to notice is that this comparison is reasonable, since as n gets large the "+1" in the series becomes insignficant compared to the "n^2", so we expect the two series to behave in essentially the same way.

(3) Often, we want to apply the comparison test but the two series don't match up quite as neatly as in the previous example. In these cases, it is easier to apply the limit comparison test (Theorem 5.12).

> Limit comparison test.
> Suppose that $0 \leq a_n, b_n$ for all n.
> (a) Suppose $\lim_{n\to\infty} a_n/b_n$ exists [and is finite]. If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
> (b) Suppose $\lim_{n\to\infty} a_n/b_n$ exists and is non-zero, or $\lim_{n\to\infty} a_n/b_n = \infty$. If $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

Let's briefly explain why (a) is true. By the assumption that $\lim_{n\to\infty} a_n/b_n$ exists (denote this limit by L), we have that $a_n \leq 2Lb_n$ for n large enough. So now apply the Comparison Test to $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2Lb_n$. So, conceptually speaking, the Limit Comparison Test is just a version of the Comparison Test, but it helps to streamline its application to particular problems.

(4) An example is the series $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$. This is similar to the series in the previous example. Again, we want to compare with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. However, the comparison test doesn't apply directly. Instead, we can use the limit comparison test. Take

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{b\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since this limit exists and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does the series $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$.

This type of comparison can be done whenever a_n is a rational function of n, where you compare it with a p-series based on the leading powers of the numerator and denominator.

(5) Next, we state the ratio test.

> Ratio test.
> Let $\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms and $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.
> (a) If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
> (b) If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
> (c) If $\rho = 1$, then the test is inconclusive.

The ratio test is essentially the comparison test with a geometric series. The idea is that you can choose a value R satisfying $\rho < R < 1$ (for case (a) showing convergence) or $1 < R < \rho$ (for case (b) showing divergence) and apply the comparison test (or limit comparison test) with the geometric series $\sum_{n=1}^{\infty} R^n$.

(6) The ratio test is useful for series involving exponential functions and factorials, since these simplify nicely in the expression $|a_{n+1}|/|a_n|$. Also, the ratio never yields any information in the case of rational functions. The ratio test allows us to handle series like

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Both of these series converge by the ratio test; see the textbook for details for some similar examples. The idea is that the exponential function 2^n in the denominator dominates the rational function n^2 in the numerator (for the first example), and the factorial function $n!$ dominates the exponential function 2^n (for the second example).

(7) Finally, we have the root test. This is also essentially the comparison test with a geometric series, though in a somewhat different way.

> Root test.
> Let $\sum_{n=1}^{\infty} a_n$ be a series and $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
> (a) If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
> (b) If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
> (c) If $\rho = 1$, then the test is inconclusive.

The root test is based on comparing with the geometric series $\sum_{n=1}^{\infty} \rho^n$. It is useful mainly in the case where a_n is of the form $a_n = b_n^n$ for some b_n. A typical example is

$$\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n},$$

which converges (since $\rho = 1/4 < 1$. It's also worth mentioning that the ratio test also works whenever the root test does. (Do you see why this is the case?)