MAT 127 LECTURE OUTLINE WEEK 7

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We are now switching to the second main unit of the course, **sequences** and **series**. This week, we will cover the basic definitions and examples.

(1) An **infinite sequence** is an ordered list of numbers $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$. A sequence can also be written with the notation $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$. The variable n is called the **index variable** and each a_n is called a **term** of the sequence. A sequence can also be thought of as a function whose domain is the positive integers, but where we write a_n in place of $a(n)$.

By convention, we usually start the index variable n at $n = 1$, but it's possible to begin with $n = 0$ or n equal to some other number. There's no real difference; the index variable n can always be chosen to start at $n = 1$ by re-indexing the sequence.

(2) There are three main ways to describe a sequence. First is to write out a list of terms, with enough terms to suggest the general pattern. For example, we might write

$$2, 4, 8, 16, 32, \ldots$$

to specify a sequence. The pattern should be evident: each term is two times the previous term. Second is to write out a **formula** for the n-th term. In the previous example, this is $a_n = 2^n$, and we can express the sequence itself as $\{2^n\}$. Third is to use a **recurrence relation**. This means the express the n-th term in terms of the previous terms, usually the $(n-1)$-th term. In the example, we would write $a_n = 2a_{n-1}$. This represents the idea that each term is obtained by doubling the previous term.

(3) There are two simple categories of sequences that appear frequently: **arithmetic sequences** and **geometric sequences**. An arithmetic sequence is one for which the <u>difference</u> between consecutive terms is constant: $a_n = a_{n-1} + k$ for some fixed number k. Converting this into a formula, an arithmetic sequence has the form

$$a_n = kn + b$$

for some numbers k, b. A typical example is the sequence

$$3, 7, 11, 15, 19, \ldots,$$

where we have $b = -1$ and $k = 4$.

A geometric sequence is one for which the <u>ratio</u> of consecutive terms is constant: $a_n = ra_{n-1}$ for some fixed constant $r \neq 0$, where r stands for 'ratio'. Note that r can be negative. The formula for a geometric sequence is of the form

$$a_n = ar^{n-1},$$

where a is the initial value and r is the ratio. A typical example is the sequence

$$2, -2/3, 2/9, -2/27, \ldots,$$

where we have $a = 2$ and $r = -1/3$.

(4) One the fundamental mathematical questions about a sequence $\{a_n\}_{n=1}^{\infty}$ is its behavior as n gets arbitrarily large. It makes sense to discuss the **limit** of a sequence the same way we talk about the limit of a function. After all, a sequence is just a particular type of function.

As a first example, consider the sequence $\{1 + 3n\} = \{4, 7, 10, 13, \ldots\}$. As n gets bigger the terms $a_n = 1 + 3n$ become arbitrarily large. We say that $a_n \to \infty$ (or $1 + 3n \to \infty$) as $n \to \infty$. We also write

$$\lim_{n\to\infty} a_n = \infty,$$

or, more concisely,

$$\lim a_n = \infty.$$

As a second example, consider the sequence $\{1 - (1/2)^n\} = \{1/2, 3/4, 7/8, \ldots\}$. We see that the terms are becoming arbitrarily close to 1 as n gets larger. Thus we write $1 - (1/2)^n \to 1$ as $n \to \infty$, or

$$\lim 1 - (1/2)^n = 1.$$

As a third example, consider the sequence $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$. This sequences oscillates between -1 and 1, and so it never approaches a single value as n larger. Thus we say that $\lim(-1)^n$ does not exist.

(5) Let's now switch to the topic of series. An **infinite series** is a sum with infinitely many terms, written as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

Interpreted literally, an infinite series involves infinitely many operations of addition, which is of course impossible to actually do on a calculator or computer. However, we can make sense of an infinite sum through the idea of limits. More precisely, we introduce the **k-th partial sum**

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k.$$

We define $\sum_{n=1}^{\infty} a_n$ to be the limit of the sequence $\{S_k\}$ of partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n.$$

If the limit exists, we say that the series **converges**. Otherwise, we say that the series **diverges**.

For example, consider the series

$$1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{n=1}^{\infty} (1/2)^{n-1}.$$

We can write out the partial sums $S_1 = 1$, $S_2 = 1 + 1/2 = 3/2$, $S_3 = 1 + 1/2 + 1/4 = 7/4$. You should notice a pattern, from which we can write S_k using the formula

$$S_k = (2^k - 1)/2^{k-1} = 2 - 1/2^{k-1}.$$

We see that $S_k \to 2$ as $k \to \infty$, and so we have

$$\sum_{n=1}^{\infty} (1/2)^{n-1} = 2.$$

(6) A famous and important example is the **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

We first want to decide whether the series converges or diverges. If it converges, we'd then like to compute the sum, if possible.

To get a feel for the series, you might try computing some partial sums. For example, $S_1 = 1$, $S_2 = 1+1/2 = 3/2$, $S_3 = 1+1/2+1/3 = 11/6$. You'll find that the partial sums appear to grow rather slowly; for example, S_{1000} is only ≈ 7.485. This might suggest that the series converges. However, the opposite is true: the harmonic series diverges, although it diverges very slowly. Here is one way to argue this (later, we'll see other methods):

Since $1/3 > 1/4$, we have $S_4 > 1 + 1/2 + 1/4 + 1/4 = 1 + 2 \cdot (1/2)$. Since $1/5, 1/6, 1/7 > 1/8$, we have $S_8 > 1 + 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 = 1+3 \cdot (1/2)$. Arguing the same way, we have $S_{2^j} > 1+j \cdot (1/2)$. But $1+j \cdot (1/2) \to \infty$ as $j \to \infty$. We conclude that the harmonic series diverges.

(7) The most important type of series is **geometric series**. This is the infinite sum of a geometric sequence covered earlier. A geometric series has the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

The value a is called the initial term, and r is called the ratio. The example in item (5) above is a geometric series.

Our goal is to determine when a geometric series is convergent and, if so, evaluate it. In order to do this, we look at the sequence of partial sums

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}.$$

There is a very nice trick for how to evaluate this. The difference $S_k - rS_k$ satisfies

$$\begin{aligned} S_k - rS_k &= (a + ar + ar^2 + \cdots + ar^{k-1}) - (ar + ar^2 + \cdots ar^{k-1} + ar^k) \\ &= a - ar^k. \end{aligned}$$

Provided that $r \neq 1$, this can be written as

$$S_k = \frac{a(1 - r^k)}{1 - r}.$$

(If $r = 1$, then it's easy to see that $S_k = ak$.) If $|r| < 1$, then the limit exists and we have

$$\sum_{k=1}^{\infty} ar^{n-1} = \lim_{k\to\infty} \frac{a(1-r^k)}{1-r} = \frac{a}{1-r}.$$

If $|r| \geq 1$, then the geometric series diverges.

(8) In many geometric series problems, you may have to do some manipulations to get the series into a form where the above formula applies. For example, consider the series

$$\sum_{n=3}^{\infty} 4\left(\frac{-1}{3}\right)^n.$$

Notice that the index n begins at 3 rather than the usual one. You have two main options in how to approach the problem, and you can do whichever is more comfortable for you. The first option is to manipulate the summation directly, while the second is to write out the series and manipulate it term-by-term. Let's do both approaches. For the first approach, we use the subsitution $m = n - 2$ to re-index the sum. This gives

$$\sum_{n=3}^{\infty} 4\left(\frac{-1}{3}\right)^n = \sum_{m=1}^{\infty} 4\left(\frac{-1}{3}\right)^{m+2} = \sum_{m=1}^{\infty} 4\left(\frac{-1}{3}\right)^3\left(\frac{-1}{3}\right)^{m-1},$$

and now we can identify $a = 4(-1/3)^3 = -4/27$ and $r = -1/3$. This gives

$$\sum_{n=3}^{\infty} 4\left(\frac{-1}{3}\right)^n = \frac{-4/27}{1+1/3} = -\frac{1}{9}.$$

For the second approach, we write out

$$\sum_{n=3}^{\infty} 4\left(\frac{-1}{3}\right)^n = -\frac{4}{27} + \frac{4}{81} - \frac{4}{3^5} + \cdots$$

For me, I find it helpful to factor out $-4/27$ to get

$$-\frac{4}{27} + \frac{4}{81} - \frac{4}{3^5} + \cdots = -\frac{4}{27}\left(1 - \frac{1}{3} + \frac{1}{9} - \cdots\right).$$

The series in parentheses on the right has leading term of 1, and so it's equal to $1/(1-r) = 1/(1+1/3) = 1/(4/3) = 3/4$. Thus the sum is $-(4/27)(3/4) = -1/9$.

(9) We end with a brief discussion of **mathematical induction**. This is a method of mathematical reasoning based on the following principle:

Principle of mathematical induction. Let $P(n)$ be a statement about the positive integers $n = 1, 2, 3, \ldots$ that may either be true or false. If:

- $P(1)$ is true;
- If $P(k)$ is true, then $P(k+1)$ is true;

then $P(n)$ is true for all positive integers n.

This principle allows us to do **proofs by induction**. Let's do an example.

Problem. Use mathematical induction to prove that

$$\sum_{n=1}^{k} n = \frac{k(k+1)}{2}.$$

According to the principle of mathematical induction, we need to check two things: the formula is true for $k = 1$ (the **base case**), and that, if the formula is true for a given k, it is also true for $k+1$ (the **inductive step**). For the base case, we have

$$\sum_{n=1}^{1} n = 1 = \frac{1(1+2)}{2},$$

so this holds.

Next, assume the formula is true for a given k. Applying this assumption, we have

$$\begin{aligned}\sum_{n=1}^{k+1} n = \sum_{n=1}^{k} n + (k+1) \quad &\text{(splitting the sum into two parts)}\\ = \frac{k(k+1)}{2} + (k+1) \quad &\text{(applying the inductive hypothesis)}\\ = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.&\end{aligned}$$

This verifies the formula for $k+1$. We conclude by induction that the formula holds for all positive integers k.