MAT 127 LECTURE OUTLINE WEEK 6

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We have our final topic related to differential equations: systems of two autonomous 1st order differential equations, which often take the form of **predator-prey equations**. Note that there is a lot more to be said about differential equations than we have time for in this class. You may eventually take Calculus 4 or a similar class focusing almost entirely on differential equations, where these and other topics are explored in much more depth.

(1) We will begin with the predator-prey equations, also called the **Lotka–Volterra equations** after the mathematicians who first studied them. Recall that we have already seen two models of population growth: exponential growth and logistic growth. Both of these deal with the population of a single species without accounting for other populations in its environment in a direct way.

Our goal now is to model two populations that interact. There are many sorts of potential interactions, but a natural one is a predator-prey relationship between two species. Let's take $R(t)$ to represent the size of a population of rabbits and $W(t)$ to represent the size of a population of wolves. We make the following assumptions: If the wolves are removed from the picture, the rabbit population will grow exponentially. Thus $\frac{dR}{dt} = kR$ for some constant $k > 0$. If the rabbits are removed from the picture, the wolves will die off at an exponential rate. Thus $\frac{dW}{dt} = -rW$ for some $r > 0$. When considered together, the rabbits are hunted by the wolves at a rate proportional to RW, the product of the two population sizes (the value RW represents the number of individual interactions between rabbits and wolves). On the other hand, the wolf population grows at a rate proportional to RW.

Combining this all together, we get the system of differential equations

$$\frac{dR}{dt} = kR - aRW$$
$$\frac{dW}{dt} = -rW + bRW,$$

where $k, r, a, b > 0$ are constants.

Take a few moments to analyze each piece of these equations. What role does each part play, and why does it make sense conceptually? When is the population of rabbits increasing or decreasing? [Answer: the rabbits are decreasing exactly when there are enough wolves for aW to be bigger than k.] When is the population of wolves increasing or decreasing? Like all models, its assumptions may be questioned and its effectiveness must be measured by its predictive power in the real world. For example, in absence of the rabbits, the wolves die off exponentially, which seems to imply that the wolves have a secondary food source other than rabbits that can partially sustain them.

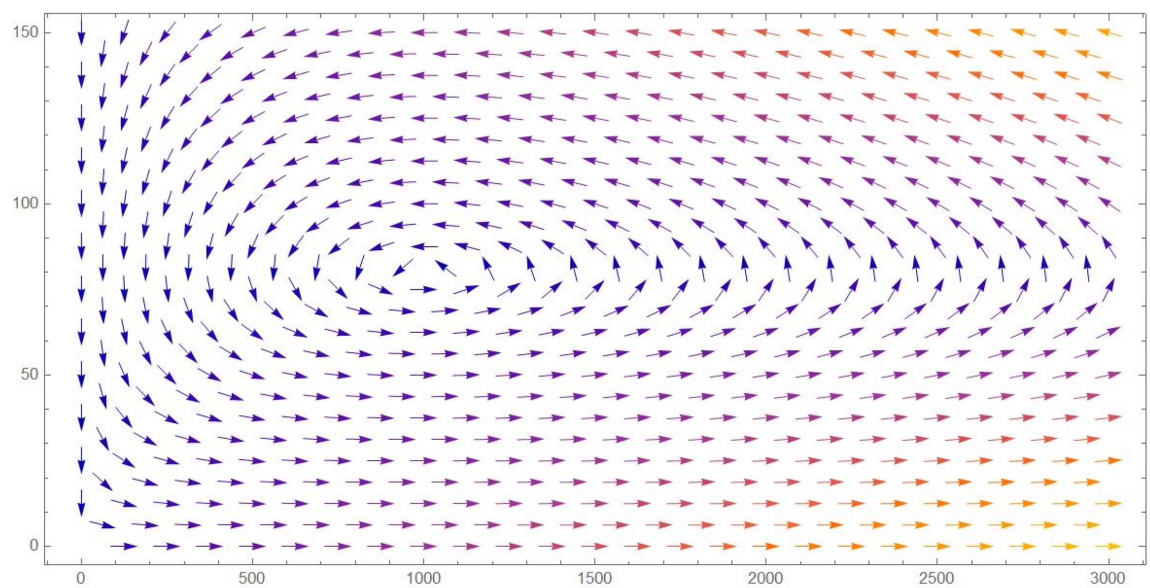
(2) Systems of differential equations such as the predator-prey equations are usually difficult or impossible to solve exactly. Instead, our goal will be a qualitative understanding of the solutions: an understanding of their long-term behavior without being concerned with the exact solution. This typically includes doing the following:
- Find the equilibrium (constant) solutions
- Plot a direction field for the system (after eliminating the variable t)
- Sketch a representative set of solutions using the direction field
- Discuss "stability" of the solutions, meaning their long-term behavior

(3) Notice that the predator-prey equations above are autonomous, meaning that they do not contain the independent variable t. This should be expected for any physical situation whose governing laws are independent of time, which is generally the case. In our rabbit/wolf situation, autonomous means that rabbits/wolves today will multiply and hunt at the same rates as they would, say, 1000 years ago. This may or may not be true in reality, but we take this as an assumption for our model.

From a mathematical point of view, the property of autonomy makes these equations easier to understand. Initially, we have three variables under consideration: the independent variable t and the dependent variables R, W. This is difficult to represent on a two-dimensional graph. However, the property of autonomy allows us to effectively ignore the time variable t and instead consider how R and W relate to each other. Applying the chain rule, we can write

$$\frac{dW}{dR} = \frac{dW/dt}{dR/dt} = \frac{-rW + bRW}{kR - aRW}.$$

This tells us how the population of wolves changes as a function of rabbits, without worrying about precisely *how long* it takes to happen. Let's plot the direction field for a typical predator-prey equation, taking $k = .08$, $r = .02$, $a = .001$, $b = .00002$ (R is represented by the horizontal axis and W by the vertical axis.



Take a moment to interpret this plot. In the bottom-right quarter of the plot, there are many rabbits and few wolves. We see that both W and R increase, since there is plenty of food for the wolves but not enough wolves yet to slow the growth of the rabbits. However, in the top-right quarter, there are enough wolves that the rabbit population begins to decrease. Eventually, in the top-left quarter, the rabbit numbers have decreased so much that there is not enough food to support the wolf population, and the wolves too begin to die off. Finally, in the bottom-left quarter, both species have become scarce enough that rabbit population again begins to grow, taking us back to the starting position.

So the solutions $(R(t), W(t))$ are going to loop around the middle point (which appears to be $(1000, 80)$) in counterclockwise fashion. In principle, these loops could be spiraling either in towards $(1000, 80)$ or away from $(1000, 80)$. However, it turns out that the solutions do not spiral; the solutions make closed loops. That is, the solutions are periodic, repeating exactly over some definite time interval. On the homework, you will be asked to mathematically verify this property.

(4) Let's complete our analysis of the predator-prey equation. To find the equilibrium solutions, we set

$$0 = \frac{dR}{dt} = kR - aRW = R(k - aW)$$
$$0 = \frac{dW}{dt} = -rW + bRW = W(-r + bR)$$

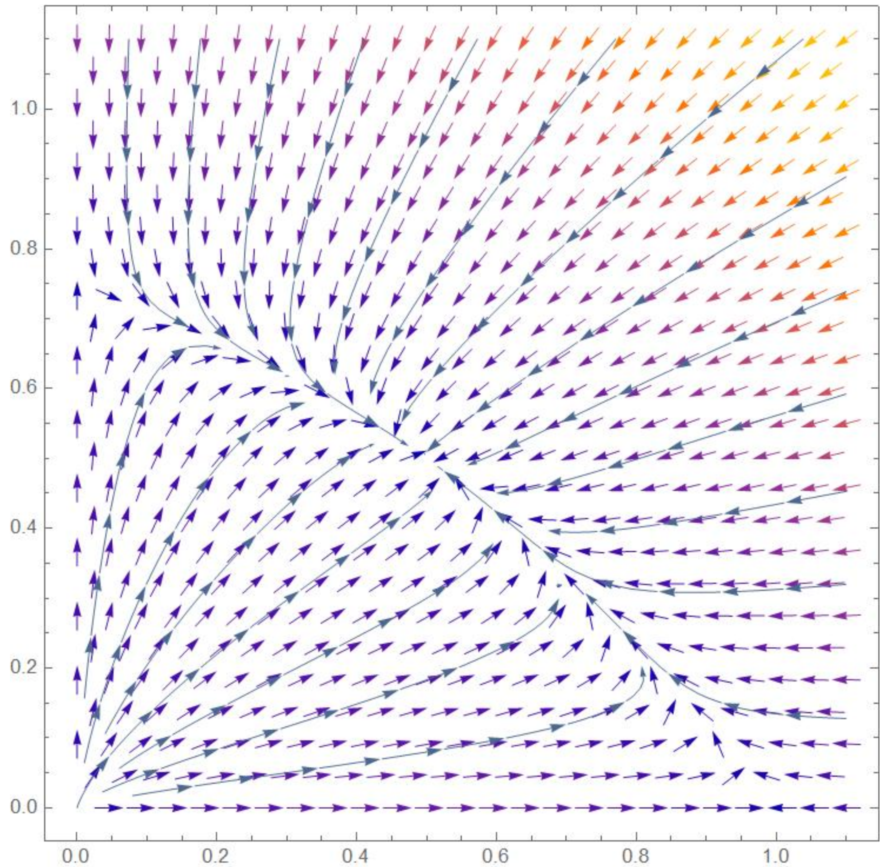
There are two possibilities: $(R, W) = (0, 0)$ and $(R, W) = (r/b, k/a)$. The first of these represents the case where both populations are non-existent in the first place. The second of these is the population sizes that are perfectly balanced: the effect of the wolves preying on rabbits exactly matches the rabbits multiplying, while the wolves can find exactly enough food to maintain their population. Using the coefficients $k = .08$, $r = .02$, $a = .001$, $b = .00002$ above, we get the equilibrium solution $(R, W) = (.02/.00002, .08/.001) = (1000, 80) = (1000, 80)$, which is the central point we saw earlier in the direction field.

The non-equilibrium solutions then circle around the non-zero equilibrium point counterclockwise, as discussed above, provided that R, W are both nonzero. For completeness, here are the other two cases. If $R = 0$, then the wolf population $W(t)$ decays exponentially to zero. If $W = 0$, then the rabbit population $R(t)$ grows exponentially indefinitely. This concludes our analysis of the predator-prey equation.

(5) We can also consider similar systems of equations modeling other interactions between species and do a similar qualitative analysis. Another situation is two species competing for the same food source. There are various possibilities. For example, the two populations can reach a stable equilibrium point where both coexist indefinitely. Or one population can drive the other to extinction. Let's take $x(t)$ to represent the number of zebra in a population, and $y(t)$ to represent the number of buffalo (in some unit such as thousands or ten thousands, so we can simplify the coefficients). In this model, we will combine <u>logistic growth</u> (the left part of the formula) with a <u>competition factor</u> (the right part of the formula) between the species. We have:

$$\frac{dx}{dt} = x(1 - x) - xy$$
$$\frac{dy}{dt} = y(.75 - y) - .5xy$$

Again, we can go through the same process of finding the equilibrium solutions and plotting the direction field. In this case, there are four equilibrium solutions, $(0, 0)$, $(0, .75)$, $(1, 0)$ and $(.5, .5)$. (Check this on your own.) Here is a plot of the direction field along with some representative solutions.

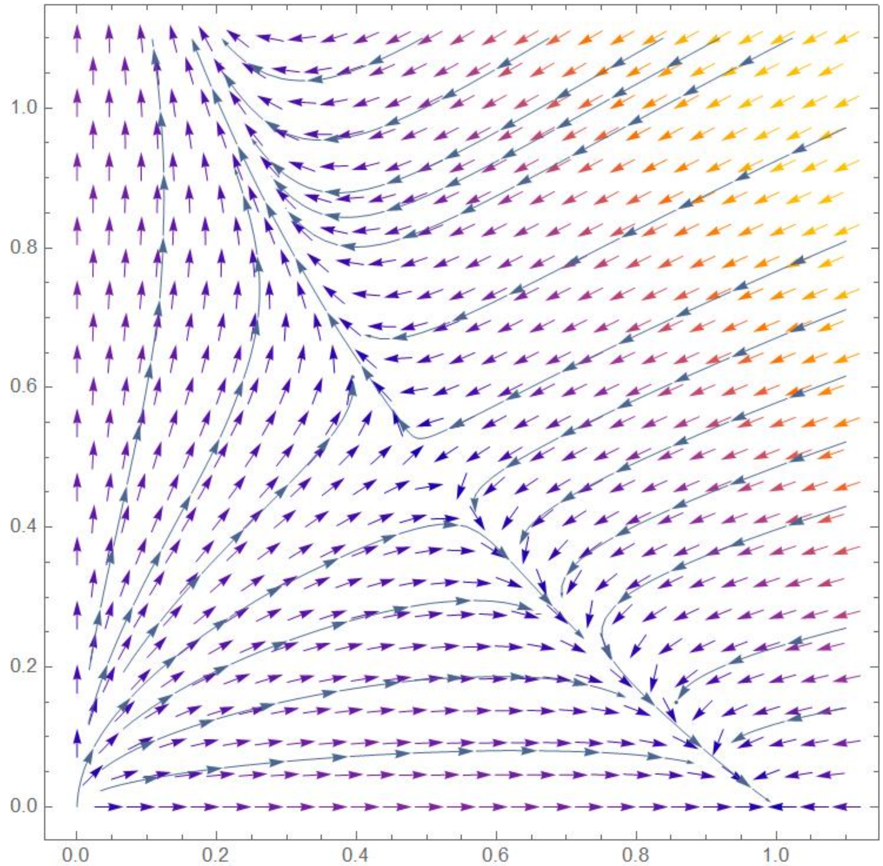


This plot gives a good idea of the long-term behavior of the solutions. If one of the populations is zero, then we see logistic growth in the other population. Otherwise, both populations are nonzero and we see that the populations will eventually approach the equilibrium point at $(.5, .5)$. Thus we have a long-term harmony between the two species. You might try to give an intuitive explanation for this behavior based upon the equations. (Maybe the two species have slightly different preferences of which plants to eat, so the ecosystem supports both species more easily than many of one species.)

(6) Now let's change the model by tweaking the coefficients:

$$\frac{dx}{dt} = x(1 - x) - xy$$
$$\frac{dy}{dt} = y(.5 - .25y) - .75xy$$

Go through the calculation to find the equilibrium points. You should get $(0, 0)$, $(1, 0)$, $(0, 2)$, $(.5, .5)$ as the four equilibrium points. Again, we plot the direction field:



This is rather different. The equilibrium point at $(.5, .5)$ in unstable. Instead, the solutions head towards either the equilibrium point $(1, 0)$ (the zebra drive the buffalo to extinction) or the equilibrium point $(0, 2)$ (the buffalo drive the zebra to extinction). In practice, the equilibrium at $(.5, .5)$ could not happen because any slight deviation would cause the populations to fall to either the 'zebra side' or the 'buffalo side'. Again, try to think of an intuitive explanation. (The competition between the two species is fiercer than competition within a species, maybe because one species spoils the food supply of the other.)