MAT 127 LECTURE OUTLINE WEEK 13

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: This is the last week of new material for the course. We will wrap up a few loose ends on the topic of power series. First is the binomial series, the last of our standard Taylor series. Next are two representative applications of power series: (1) computing an integral using a power series representation where a closed form solution is not possible, and (2) finding a numerical solution to a differential equation.

(1) Let's set the stage with a review of a non-calculus topic: expanding a polynomial of the form $(x+y)^r$. We get the table here:

$$\begin{array}{rc}
(x+y)^0 = & 1 \\
(x+y)^1 = & x+y \\
(x+y)^2 = & x^2+2xy+y^2 \\
(x+y)^3 = & x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 = & x^4+4x^3y+6x^2y^2+4xy^3+y^4 \\
(x+y)^5 = & x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5 \\
\vdots & \vdots
\end{array}$$

The coefficients that show up in the expansion are the same ones that form the famous *Pascal's Triangle*:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\vdots
\end{array}$$

The rule for forming Pascal's triangle is that each number is the sum of the two numbers above it. For example, the 10 in the bottom row is equal to $4+6$. Pascal's triangle is an interesting object with many nice properties (about which many online videos can tell you about). The key thing is that the entries of the triangle are given by the *binomial coefficient*

$$\binom{r}{n} = \frac{r!}{n!(r-n)!},$$

where r is the row of the triangle and n (where $0 \le n \le r$) is the column of the triangle. Take a moment to check that the values in Pascal's triangle match the formula.

If we take $y = 1$ in the first table, we get the Taylor series for the function $f(x) = (1+x)^r$. For example, $(1+x)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$. Note that this doesn't require calculus, though.

(2) We want to use the same idea to find a Taylor series (at $a = 0$) for the function $f(x) = (1+x)^r$, where r is any real number. For example, if $r = 1/2$, we have the function $f(x) = \sqrt{1+x}$. This can be done directly from the Taylor series. Assume $r \neq 0, 1, 2, \ldots$, since we covered that case in item (1). We have

$$\begin{array}{ll} f(x) = (1+x)^r & f(0) = 1 \\ f'(x) = r(1+x)^{r-1} & f'(0) = r \\ f''(x) = r(r-1)(1+x)^{r-1} & f''(0) = r(r-1) \\ \quad\vdots & \quad\vdots \\ f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) = r(r-1)\cdots(r-n+1) \end{array}$$

We can extend the definition of the binomial coefficient to all r by the formula

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

Note that this is consistent with the previous formula if $r = 0, 1, 2, \ldots$.

In this notation, we can write out the series

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$

This series converges for $|x| < 1$, as can be shown by the ratio test. (The convergence at the endpoints $|x| = 1$ depends on r; see the textbook for discussion. If $r = 0, 1, 2, \ldots$, then the series terminates after r terms, and in particular the series converges for all x.)

(3) Next, we turn to our two applications of power series. The first concerns functions like $f(x) = e^{-x^2}$ and $f(x) = \cos(x^2)$ that do not have antiderivatives that can be expressed as formulas using standard functions. Nevertheless, these functions can be integrated using series, which is perfectly fine for numerical approximations. Especially important is the function $f(x) = e^{-x^2}$, which is used often in statistics as the *normal distribution* or *bell curve*. (In statistics, this function appears with additional constants to represent the *mean* and *standard deviation* of the distribution, but we'll keep things simple here.)

The function $f(x) = e^{-x^2}$, as we noted, does not have a closed-form antiderivative. However, by writing it as the series

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots,$$

we can find a power series for the antiderivative:

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}.$$

We can then use this to numerically compute definite integrals. For example:

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}\bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

In a typical statistics book, you'll find an appendix with a table of probabilities for a normally distributed set of data. This would tell you, say, the probability that an object lies between .5 and 1.5 standard deviations of the mean. This is exactly the integral $\displaystyle\int_{.5}^{1.5} e^{-x^2/2}\,dx$, which you're now equipped to handle. With a little patience, you could produce such a table of probabilities yourself.

(4) The second application relates to solving differential equations using series. This topic is a nice one to end on, since it takes us full circle to the first unit of our course on differential equations. As we remarked then, most differential equations cannot be solved exactly. Instead, one has to rely on numerical methods. Power series provided such a method. Let's illustrate this with an example (see Example 6.21 in your book).

Consider the equation $y'' - xy = 0$ with initial conditions $y(0) = a$ and $y'(0) = b$. The solution is some function $y(x)$. The idea is to assume that $y(x)$ has a series representation $y(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ and substitute this into the differential equation. First we need to take two derivatives: $y'(x) = \displaystyle\sum_{n=1}^{\infty} nc_n x^{n-1}$ and $y''(x) = \displaystyle\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Subsituting gives

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \cdot \sum_{n=0}^{\infty} c_n x^n = 0.$$

To make this visually easier to digest, let's put in the form

$$2 \cdot 1c_2 + (3 \cdot 2)c_3 x + (4 \cdot 3)c_4 x^2 + (5 \cdot 4)c_5 x^3 \cdots = c_0 x + c_1 x^2 + c_2 x^3 + \cdots .$$

The two sides are equal as series if and only if all the coefficients match. This gives a series of algebraic relations that allow us to find the coefficients $c_0, c_1, c_2, \ldots$ recursively:

$$\begin{aligned} 2 \cdot 1c_2 &= 0 \\ 3 \cdot 2c_3 &= c_0 \\ (4 \cdot 3)c_4 &= c_1 \\ (5 \cdot 4)c_5 &= c_2 \\ &\vdots \end{aligned}$$

The general relation is $n(n-1)c_n = c_{n-3}$, or equivalently $c_n = c_{n-3}/(n(n-1))$.

Now apply the initial conditions: we have $f(0) = c_0 = a$ and $f'(0) = c_1 = b$. Putting this together, we have

$$\begin{aligned}
c_0 &= a \\
c_1 &= b \\
c_2 &= 0 \\
c_3 &= a/(3 \cdot 2) \\
c_4 &= b/(4 \cdot 3) \\
c_5 &= 0 \\
c_6 &= a/(6 \cdot 5 \cdot 3 \cdot 2) \\
c_7 &= b/(7 \cdot 6 \cdot 4 \cdot 3) \\
c_8 &= 0 \\
&\vdots
\end{aligned}$$

and so $y(x) = a + bx + \frac{a}{3 \cdot 2}x^3 + \frac{b}{4 \cdot 3}x^4 + \frac{a}{6 \cdot 5 \cdot 3 \cdot 2}x^6 + \frac{b}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots$.