MAT 127 LECTURE OUTLINE WEEK 12

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We will expand on our study of power series from last week. The main topic is a particular type of power series called a **Taylor series**, which is defined at each point of a smooth (i.e., infinitely many times differentiable) function.

(1) Let's start with the main definition. For any smooth function $f(x)$, we define the **Taylor series of $f(x)$ centered at $x = a$** to be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots .$$

To clarify the notation: $f^{(n)}$ is the n-th order derivative of f, i.e., the function f differentiated n times. We follow the convention that $0! = 1$ and $(x-a)^0 = 1$, even for $x = a$. (Recall that usually the expression 0^0 is considered indeterminate.) Also, $f^{(0)}(x) = f(x)$.

Denote for now the Taylor series of $f(x)$ by $P(x)$. Observe first that $P(x)$ is a perfectly good power series, so everything we discussed last week applies (radius and interval of convergence, algebraic operations, differentiation and integration). To understand the formula for $P(x)$, let's compute its derivatives. The first derivative is

$$P'(x) = f'(x) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots .$$

In particular, $P'(a) = f'(a)$. Continuing on, the second derivative is

$$P''(x) = f''(a) + f'''(a)(x-a) + \frac{f^{(4)}(a)}{2!}(x-a)^2 + \cdots .$$

In particular, $P''(a) = f''(a)$. In a similar manner, we see that $P^{(n)}(a) = f^{(n)}(a)$ for all n. In other words, $P(x)$ is a power series whose derivatives of all orders match that of $f(x)$ at the point $x = a$.

For this reason, it is natural to expect that in fact $f(x) = P(x)$ for all x, at least in the interval of convergence for the power series. That is, for all points $x = a$, the function $f(x)$ is equal to its Taylor series centered at $x = a$ in some interval containing a. This is true for "nice enough" functions, such as e^x and $\cos(x)$ and $\ln(1+x)$ and so forth. We will return to this idea later this week with Taylor's Remainder Theorem.

However, it turns out that not every smooth function is equal to its Taylor series. In the homework problems, you get to explore an example of such a function.

(2) Let's practice using the Taylor series formula. Our first example is the exponential function.

Example. Find the Taylor series for $f(x) = e^x$ centered at 0.

The basic idea is to compute derivatives $f'(0)$, $f''(0)$, $f'''(0)$ and so forth and look for a pattern. In this case, it is easy: $f^{(n)}(x) = e^x$ for all n, and thus $f^{(n)}(0) = 1$. Substituting this into the formula, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \cdots .$$

This is the power series representation for e^x. In fact, this is an especially nice series: let's apply the ratio test to find the radius of convergence. For each x, we have

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since $\rho = 0 < 1$, we conclude by the ratio test that the series converges for all x. That is, the radius of convergence is ∞ and the interval of convergence is $-\infty < x < \infty$.

(3) A good second example is $f(x) = \sin(x)$. The derivatives are slightly more involved, but we still land on a nice pattern. We have

$$\begin{array}{ll} f(x) = \sin(x) & f(0) = 0 \\ f'(x) = \cos(x) & f'(0) = 1 \\ f''(x) = -\sin(x) & f''(0) = 0 \\ f'''(x) = -\cos(x) & f'''(0) = -1, \end{array}$$

after which the series falls into a repeating pattern every four derivatives (i.e., $f^{(n+4)}(x) = f^{(n)}(x)$. Notice that $f^{(n)}(0)$ is non-zero exactly when n is odd, and these values then alternate between 1 and -1. The Taylor series for $f(x) = \sin(x)$ can be written as

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Notice how this summation skips over the even powers of x.

Doing a similar procedure gives a series for $\cos(x)$:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

The radius of convergence for both of these series is ∞. This can be shown by the ratio test, as we did for the series for e^x.

(4) Once we've produced Taylor series for the standard functions (e^x, $\sin(x)$, etc.), we can find series for other, potentially more complicated functions. This is often done by substitution, for example,

$$e^{-2x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{3n}}{n!},$$

or some algebraic manipulations. Or you might go the other direction: given a Taylor series, you are asked to identify the function that it came from.

(5) In the examples so far, our series have been centered at $a = 0$. We can also find Taylor series with other centers, and in some cases this may be advantageous.

Example. Find the Taylor series for the polynomial $f(x) = x^3 - x + 1$ centered at $a = -1$.

Let's apply the formula for Taylor series directly. Compute derivatives: $f'(x) = 3x^2 - 1$, $f''(x) = 6x$, $f'''(x) = 6$ and $f^{(n)}(x) = 0$ for $n \geq 4$. Then $f(-1) = 1$, $f'(-1) = 2$, $f''(-1) = -6$, $f'''(-1) = 6$, and the Taylor series formula gives

$$\begin{aligned} f(x) &= 1 + 2(x+1) - \frac{6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3 \\ &= 1 + 2(x+1) - 3(x+1)^2 + (x+1)^3 \end{aligned}$$

Note that we really have exactly the same polynomial as we started with, but just written in a different form. If we were to expand out the expression $1 + 2(x+1) - 3(x+1)^2 + (x+1)^3$, we would get exactly the polynomial we began with.

In particular, the radius of convergence for the series $1+2(x+1)-3(x+1)^2+(x+1)^3$ is ∞, since it's just a polynomial equal to the original one. The series must converge because it only has finitely many terms.

(6) Let's give another illustrative example to this effect. The function $f(x) = 1/(1-x)$ has the standard Taylor series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

as should be by now familiar. What if we want to find a Taylor series for the same function with the new center $a = -1$? This can be done with some clever algebraic manipulation (the idea is to isolate an "$x+1$" in the formula):

$$\begin{aligned} \frac{1}{1-x} &= \frac{1}{2-(x+1)} = \frac{1}{2}\left(\frac{1}{1-(x+1)/2}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}}. \end{aligned}$$

This might seem like a contrived problem, but our new series does have one advantage over the standard one: its radius of convergence is 2 rather than 1, so the series converges to $1/(1-x)$ on a much larger interval.

(7) A final subtopic is Taylor's Remainder Theorem. As the name suggests, this gives an estimate on the difference between the **k-th Taylor polynomial**

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and the actual value $f(x)$ of the function at x. Moreover, this theorem partially answers the question of deciding when a Taylor series equals to function it came from.

For the purposes of this class, we will state the theorem without discussing the underlying idea in depth. If you're curious, the textbook gives a proof. But you can simply think of it as a sort of extension of the Mean Value Theorem (which deals with *first* derivatives).

Taylor's Remainder Theorem.
Let $f(x)$ be a smooth function defined on an interval containing the point a. Define the k-th remainder as

$$R_k(x) = f(x) - p_k(x) = f(x) - \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Fix a point b. Assume there is a constant $M > 0$ such that $|f^{(k+1)}(x)| \leq M$ for all x between a and b. Then

$$R_k(x) \leq \frac{M}{(k+1)!}|x-a|^{k+1}$$

for all x between a and b.

We'll give a simple illustration of this theorem. For the function $f(x) = \sin(x)$, we have $|f^{(k+1)}(x)| \leq 1$ for all x and all k. According to this theorem, $R_k(x) \leq |x|^{k+1}/(k+1)!$ for all x. In particular, $R_k(x) \to 0$ as $k \to \infty$ for all x. This justifies the statement that $\sin(x)$ really is equal to its Taylor series.