MAT 127 LECTURE OUTLINE WEEK 11

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We now enter the final unit of the course: **power series**. This continues our study of series from the previous chapter. The new idea is to consider series that contain a variable x and therefore are functions of x.

(1) We will begin to look at **power series**. Let's first give some motivation for the topic. Imagine that you need to program a computer to evaluate some relatively complicated function, say the sine function or natural logarithm function. Computers are very adept at simple operations like addition and multiplication. So the question is: can you program a computer to evaluate $\sin(x)$ or $\ln(x)$ (within some small error) just by using addition and multiplication? The answer is yes, and power series provide a way to do it.

(2) A **power series (centered at** 0**)** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + x_2 x^2 + \cdots .$$

A power series resembles a geometric series. In fact, if you take $1 = c_0 = c_1 = c_2 = \cdots$, then you have the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots ,$$

which is the geometric series with ratio $r = x$. Recall that this series converges if $-1 < x < 1$ and diverges otherwise.

More generally, a **power series (centered at** a**)** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

A power series usually converges for some values of x and diverges for others. Note that all power series converge if $x = a$, since then it evaluates to c_0.

(3) This leads to a general theorem about when a power series converges.

Theorem. For any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, one of the following three possibilities holds:

(i) The series converges for $x = a$, and diverges for all $x \neq a$.
(ii) The series converges for all x.
(iii) There is a value $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. [When $|x-a| = R$, the series may either converge or diverge.]

Let's explain the idea of the theorem. Assume for simplicity that $a = 0$. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges for some value $x = d$. We claim that $\sum_{n=0}^{\infty} c_n x^n$ must converge whenever $|x| < |d|$. We see this by writing $|c_n x^n|$ as $|c_n d^n||x/d|^n$. If n is large, then $|c_n d^n| \leq 1$, since we assumed that the series converges for $x = d$. But then we have $|c_n x^n| \leq |x/d|^n$, where $|x/d| < 1$. Now we apply the (limit) comparison test with the convergent geometric series $\sum_{n=0}^{\infty} |x/d|^n$ to conclude that $\sum_{n=0}^{\infty} c_n x^n$ also converges. This argument justifies why the set of points for which the series converges must be an interval centered at a, as opposed to some more complicated set.

(4) The value R in the previous theorem is called the **radius of convergence**. The set of all values x for which $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is called the **interval of converence**. If $0 < R < \infty$, then there are four possibilities: the closed interval $[a-R, a+R]$, the open interval $(a-R, a+R)$, and the half-open intervals $(a-R, a+R]$ and $[a-R, a+R)$. (See part (iii) of the previous theorem.)

(5) Here is a standard problem: Given a power series, find its interval and radius of convergence. See Example 6.1 in the book for some examples, such as:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sum_{n=0}^{\infty} n! x^n, \quad \sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}.$$

Here's a handy tip: you can always use the ratio test to find the radius of convergence. You then have to test the two endpoints $x = a - R$ and $x = a + R$ separately for convergence.

(6) As mentioned in the first item above, the motivation of power series is to find a way to represent complicated functions in terms of simple addition and multiplication. The formula for the sum of a geometric series gives our first example of this:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots . \tag{1}$$

That is, the function $f(x) = 1/(1-x)$ is represented by the series on the right in (1). There is one difference to keep in mind: the function $f(x) = 1/(1-x)$ is defined for all $x \neq 1$. On the other hand, the geometric series converges if and only if the ratio has absolute value less than 1, i.e., if $|x| < 1$. In other words, the radius of convergence of the series is 1, and the interval of convergence is $-1 < x < 1$. So the power series representation of a function is usually *local* rather than *global*.

(7) The previous example might seem like a relatively unimportant function. However, from just the one series (1) we can obtain a large number of other series. For example, replacing "x" with "$-x$" in (1) gives

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots .$$

Similarly, replacing "x" with "$-x^2$" gives

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots .$$

(8) One nice feature of power series is that they behave well under addition, multiplication, differentiation and integration. In particular, it is mathematically correct to integrate and differentiate power series term-by-term. For example, we can take the derivative of both sides of (1) to get

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty}\frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

The previous series can be rewritten as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

This series can be differentiated again to give (with some algebra) power series for $1/(1-x)^n$ for all n. In fact, using partial fraction decomposition and some algebra, we can come up with a representation for any rational function using this approach.

(9) In the other direction, we can integrate the power series for $1/(1+x)$ to get

$$\ln(1+x) = C + \sum_{n=0}^{\infty}\int(-1)^n x^n\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$$

for some C. Since $\ln(1+0) = 0$, we see that $C = 0$. After reindexing the previous series, we finally have

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

[The series, along with many others, can be found in the table on p. 585 in the book. However, the book has a typo since it has "$n = 0$" in place of "$n = 1$".] In a similar way, we can integrate the series for $1/(1+x^2)$ to get a power series for $\arctan(x)$:

$$\arctan(x) = \int\frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty}\int(-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

(10) Some final remarks: when you differentiate or integrate a power series, the radius of convergence does not change. For example, the radius of convergence for the series for $\ln(1+x)$ and $\arctan(x)$ are both 1. However, convergence at the endpoints might be affected. So this must be inspected separately if you need to find the interval of convergence for such a series.

Already, we can find a power series representation for many functions. However, we haven't yet done functions like e^x and $\sin(x)$. There is a general method to find the power series representation of any smooth function. We will cover this in Week 12.