MAT 127 LECTURE OUTLINE WEEK 10

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We have our final topic in the chapter: **alternating series**. This is a series whose terms alternate between being positive and negative. Fortunately, the convergence of alternating series is usually simple, since the positive and negative terms mostly cancel and thus make it easier for the series to converge.

(1) This is a good moment to recall a fact about *sequences* that we glossed over before: the monotone convergence theorem. First, let's state the terminology. A sequence $\{a_n\}$ is **increasing** if $a_n \leq a_{n+1}$ for all n. A sequence $\{a_n\}$ is **decreasing** if $a_n \geq a_{n+1}$ for all n. A sequence is **monotone** if it is either increasing or decreasing. Finally, a sequence is **bounded** if it there is a value $M > 0$ such that $|a_n| \leq M$ for all n.

> **Monotone convergence theorem.** Any bounded, monotone sequence converges.

Let's discuss the idea of the theorem. Assume for simplicity that $\{a_n\}$ is increasing. Since the sequence is increasing, the terms a_n will either continue to grow by a definite amount (i.e., a_n goes to infinity), or the a_n will flatten out (i.e., approach a finite limit). However, since $\{a_n\}$ is bounded, the first of these possibilities is ruled out. So the sequence must converge to a limit.

(2) As mentioned above, an alternating series in one whose terms alternate between being positive and negative. An alternating series has the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n}b_n,$$

where each $b_n \geq 0$. An example is

$$\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots.$$

(3) Here is the main fact about alternating series:

> **Alternating series test.**
> The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$ converges if
> (i) $0 \leq b_{n+1} \leq b_n$ for all n
> (ii) $\lim_{n\to\infty} b_n = 0$.
>
> Moreover, in this case we have the remainder estimate $|R_N| \leq b_{N+1}$.

Take a moment to look at Figures 5.17 and 5.18 in the book, which should help illustrate the idea of the theorem. Namely, the sequence S_k of partial sums of an alternating series bounce up and down with n. But since $b_{n+1} \leq b_n$, these bounces get smaller and smaller. What we have is that the sequence $\{S_{2k}\}$ of even partial sums is monotone (either increasing or decreasing), as is the sequence $\{S_{2k+1}\}$ of odd partial sums (in the opposite direction as $\{S_{2k}\}$). So both $\{S_{2k}\}$ and $\{S_{2k+1}\}$ are bounded, monotone sequences. By the Monotone Convergence Theorem, they both converge. Using property (ii) that $\lim_{n\to\infty} b_n = 0$, they both must converge to the same limit.

(4) There is one example that best illustrates the alternating series test. This is the **alternating harmonic series**

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Check that the alternating series test applies. Therefore, the alternating harmonic series converges. This is in contrast to the usually harmonic series, which diverges.

(5) The previous example motivates one final piece of terminology. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\displaystyle\sum_{n-1}^{\infty} |a_n|$ converges. A series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges conditionally** if if converges but $\displaystyle\sum_{n-1}^{\infty} |a_n|$ diverges. So the alternating harmonic series converges conditionally.

Absolute convergence is a stronger type of convergence. We'll state the relevant theorem:

Absolute convergence implies conditional convergence.
If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges.

There is a nice proof for this theorem. The idea is to let $b_n = |a_n| - a_n$, which satisfies $0 \leq b_n \leq 2|a_n|$. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then we can use the comparison test to conclude that $\displaystyle\sum_{n=1}^{\infty} b_n$ also converges. We then use some algebra to conclude that $\sum_{n=1}^{\infty} a_n$ also converges. See p. 501 in the book for more details.