⇒ color chalk ⇐

Last time: Integral Test for Convergence of Σ

If f = positive, decreasing, continuous on $[1,\infty)$,
$\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_{1}^{\infty} f(x)\,dx$ converges

Applicable only to $\sum_{n=1} a_n$ with $a_n > 0$ for all n (> some N)

Warning: $\sum_{n=1}^{\infty} f(n) \neq \int_1^{\infty} f(x)\,dx$ if converges; $\geq$ (area over-estimated)

Main corollary: p-series test

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1}$$

Estimate $\sum_{n=1}^{\infty} f(n)$ with f = positive, decreasing, continuous on $[1,\infty)$

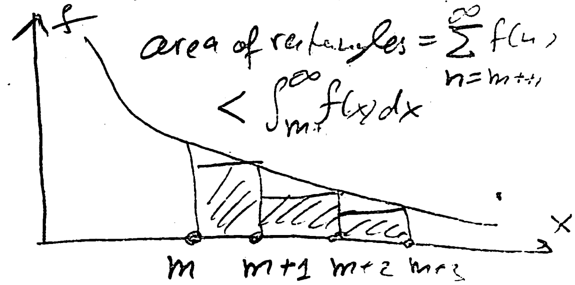
by $S_m = \sum_{n=1}^{n=m} f(n) \leftarrow$ finite sum

$$\boxed{\int_{m+1}^{\infty} f(x)\,dx \overset{(a)}{<} \sum_{n=m+1}^{\infty} f(n) = \underbrace{\sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{n=m} f(n)}_{\text{error}} \overset{(b)}{<} \int_m^{\infty} f(x)\,dx}$$

(b) and "$\int f(x)\,dx$ converges" $\Longrightarrow$ "$\sum_{n=1}^{\infty} f(n)$" converges

b/c of underestimating area under the graph of f

area of rectangles $= \sum_{n=m+1}^{\infty} f(n) < \int_m^{\infty} f(x)\,dx$



Next: Comparison Test

(1) If $0 \leq a_n \leq b_n$ for all n ($\geq$ some N) and $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$

(2) If $0 \leq b_n \leq a_n$ for all n ($\geq$ some N) and $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$

Substance: Σ with smaller positive terms is more likely to converge

(1) $\Longleftrightarrow$ (2) with roles of $\{a_n\}$ and $\{b_n\}$ switched

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Example 1: $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converge/diverge?

Try $0 \leq \frac{1}{n^2+n} \leq \frac{1}{n}$? No!

$0 \leq \frac{1}{n^2+n} \leq \frac{1}{n^2}$ for all n

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series test)

$$\Longrightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+n} \text{ converges}}$$

Another way: integral test

same answer as $\int_1^{\infty} \frac{1}{x^2+x}\,dx$ b/c $f(x) = \frac{1}{x^2+x} > 0$ decreasing, continuous on $[1,\infty)$

$$\frac{1}{(x+0)(x+1)} = \frac{1}{1-0}\left(\frac{1}{x+0} - \frac{1}{x+1}\right) = \frac{1}{x} - \frac{1}{x+1}$$

$$\int_1^{\infty} \frac{dx}{x^2+x} = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \left(\ln(x) - \ln(x+1)\right)\Big|_1^{\infty}$$

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$(\ln x - \ln(x+1))\Big|_1^\infty = \ln \frac{x}{x+1}\Big|_1^\infty$

$= \lim_{x\to\infty} \ln\frac{1}{1+1/x} - \ln\frac{1}{1+1} = \underbrace{\ln\frac{1}{1+0}}_{0} \underbrace{- \ln\frac{1}{2}}_{+\ln 2}$

$\therefore \int_1^\infty \frac{1}{x^2+x}\,dx$ converges $\Rightarrow \sum_{n=1}^\infty \frac{1}{n^2+n}$ converges

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3rd way: directly via partial sums $S_n = \sum_{k=1}^{k=n} \frac{1}{k(k+1)}$ and partial fractions

$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$S_n = \sum_{k=1}^{k=n} \frac{1}{k(k+1)} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= \frac{1}{1} - \frac{1}{n+1} \longrightarrow 1$ as $n \to \infty$

$\therefore \sum_{n=1}^\infty \frac{1}{n(n+1)}$ converges

and $= 1$ (also computed Σ!)

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Example 2: $\sum_{n=1}^\infty \frac{\ln n}{n}$ converge/diverge?

$0 \le \frac{1}{n} \le \frac{\ln n}{n}$ for all $n \ge 3$ ($\Rightarrow \ln n \ge 1$)

$\sum_{n=1}^\infty \frac{1}{n}$ diverges $\Longrightarrow \sum_{n=1}^\infty \frac{\ln n}{n}$ diverges

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Another way: Integral Test

$f(x) = \frac{\ln x}{x} > 0$ if $x > 1$, continuous

$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} < 0$ if $\ln x > 1$

$\Longrightarrow f = f(x)$ decreasing if $x \ge e$

$\int_3^{+\infty} \frac{\ln x}{x}\,dx = \int_{\ln 3}^{\ln\infty} u\,du = \frac{1}{2}u^2\Big|_{\ln 3}^{\infty} = \frac{1}{2}\big(\underbrace{\infty^2}_{\infty} - (\ln 3)^2\big)$

$u = \ln x,\ du = \frac{dx}{x}$

$\therefore \sum_{n=1}^\infty \frac{\ln n}{n}$ diverges

Example 2: $\sum_{n=1}^\infty \frac{1}{2^n - 1}$ converge/diverge

almost like $\sum_{n=1}^\infty \left(\frac{1}{2}\right)^n$ ← converges

$0 \le \left(\frac{1}{2}\right)^n \le \frac{1}{2^n - 1} \not\Longrightarrow \sum_{n=1}^\infty \frac{1}{2^n-1}$

No: need $\frac{1}{2^n-1} \le$ something convergent

$2^n - 1 = 2^{n-1} + 2^{n-1} - 1 \ge 2^{n-1} \Longrightarrow \frac{1}{2^n - 1} \le \frac{1}{2^{n-1}}$

$\sum_{n=1}^\infty \frac{1}{2^{n-1}}$ converges $\Longrightarrow \sum_{n=1}^\infty \frac{1}{2^n - 1}$ converges

over ⟹

Better: Limit Comparison ("~~looks~~ like") $0 \le a_n < Cb_n$ Test for Convergence of Series

(1) If $b_n > 0$ for all n ($\ge$ some N), $\sum_{n=1}^{\infty} b_n$ converges, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then $\sum_{n=1}^{\infty} a_n$ converges

(2) If $a_n > 0$ for all n ($\ge$ some N), $\sum_{n=1}^{\infty} b_n$ diverges and $\lim_{n\to\infty} \frac{b_n}{a_n}$ exists, then $\sum_{n=1}^{\infty} a_n$ diverges

Motivation: $\lim_{n\to\infty} \frac{a_n}{b_n} = c \Rightarrow a_n \approx c\, b_n$ as $n \to \infty$

$\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n \approx c \sum_{n=1}^{\infty} b_n$ converges

$\Longleftarrow$ if $c \neq 0$

(1) $\Leftrightarrow$ (2) with roles of $\{a_n\}$ and $\{b_n\}$ switched

Example 3': $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge/diverge?

$0 < b_n = \frac{1}{2^n}$, $\sum_{n=1}^{\infty} b_n$ converges

$$\lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{2^n/2^n}{2^n/2^n - 1/2^n}$$

$$= \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = \frac{1}{1-0} = 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n-1}$ also converges

Example 4: $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converge/diverge?

$0 < b_n = \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$$\lim_{n\to\infty} \frac{2/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = \lim_{n\to\infty} \frac{2n^2/n^2}{n^2/n^2 - 1/n^2}$$

$$= \lim_{n\to\infty} \frac{2}{1 - 1/n^2} = \frac{2}{1-0} = 2$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2-1}$ also converges

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Previously:

used Integral Test

Partial Fractions to get sequence of partial sums, telescoping cancelation ... exact li...

Tu: used integral test for $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

Th: looked at partial sums ~~for~~ ~~$\sum$~~ // all with same result

Warning: in Limit Comparison Test (1), "$b_n > 0$" is necessary.

E.g. $a_n = \frac{1}{n}$, $b_n = \frac{(-1)^n}{\sqrt{n}}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n}{(-1)^n/\sqrt{n}} =$$

$$= \lim_{n\to\infty} (-1)^n \underbrace{\frac{\sqrt{n}}{n}}_{= \frac{1}{\sqrt{n}}} = 0$$

Next week: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$ converges by Alternating Series Test

but $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test