MAT 127 ⇒ color chalk ⇐

03/24/22

#2

Sequences, series, and sequences of partial sums

- sequence $\{a_n\}_{n\geq 1} = a_1, a_2, a_3, \ldots$
- series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$
- partial sums: $s_n = \sum_{k=1}^{k=n} a_k = a_1 + a_2 + \ldots + a_n$

$\sum_{n=1}^{\infty} a_n$ converges/diverges if $\{s_n\}$ does $= \lim_{n\to\infty} s_n$ ← not a_n!

Geometric series $c, cr, cr^2, \ldots$

- converges to $\frac{c}{1-r}$ if $|r|<1$
- diverges if $|r| \geq 1$

Th: used to express $2.2\overline{36} = 2.2363636\ldots$ as simple fraction: p/q p, q = integers

$\sum_{n=1}^{\infty} (b_n - b_{n+m})$ $\{b_n\}$ sequence, $m \geq 1$

converges if $\{b_n\}$ converges

Can use telescoping cancellation if $\lim_{n\to\infty} b_n = 0$

$\hookrightarrow \sum_{n=1}^{\infty} (b_n - b_{n+m}) = \sum_{n=1}^{n=m} b_n$, e.g. $b_n = \frac{1}{n}$

if $\lim_{n\to\infty} b_n = b_\infty \neq 0$, get correction of $-m b_\infty$

test converge/diverge for $\sum_{n=1}^{\infty} (b_n - b_{n+m})$ via

Most important divergence test

if $\{a_n\}$ diverges or $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges

Warning: if $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ may converge or diverge

∴ one-sided test ⟹ Reason, Exam

~~direct~~ Two-sided test: **Integral Test**

If $f = f(x)$ is a positive, decreasing, (continuous) function on $[1, \infty)$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ exists

Example 1: $\sum_{n=1}^{\infty} \frac{1}{n}$ converge/diverge?

same as $\int_1^{\infty} \frac{1}{x}dx = \ln x \Big|_1^{\infty} = \underbrace{\ln \infty}_{\infty} - \underbrace{\ln 1}_{0}$

∴ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Harmonic Series. done last time directly

Example 2: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge/diverge

same as $\int_1^{\infty} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^{\infty} = -\left(\frac{1}{\infty} - \frac{1}{1}\right) = 1$

∴ $\int_1^{\infty} \frac{1}{x^2}dx$ converges

Later $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \simeq 1.64$

$s_1 = \frac{1}{1} = 1$ $s_2 = \frac{1}{1} + \frac{1}{2^2} = \frac{5}{4} = 1.25$ $s_3 = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36} \simeq 1.36$

Example 3: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge/diverge

$p \leq 0 \rightarrow$ diverges

$p = 1 \rightarrow$ diverges (Example 1)

$p \neq 1$, $p > 0$: same answer as for

$\int_1^{\infty} \frac{dx}{x^p} = -\frac{1}{(p-1)} x^{-(p-1)}\Big|_1^{\infty} = -\frac{1}{p-1}\left(\underbrace{\infty^{-(p-1)}}_{0 \text{ if } p>1,\ \infty \text{ if } p<1} - 1\right)$

∴ converges if $p > 1$, diverges if $p \leq 1$

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Major Corollary of Integral Test for Σ:

p-series test/thm:

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p>1}$$

E.g. $\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}, \sum_{n=1}^{\infty} \frac{1}{n^2}, \sum_{n=1}^{\infty} \frac{1}{n^3}$ converse

$\sum_{n=1}^{\infty} \frac{1}{n}, \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverse

Example 4: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge/diverge

same as $\int_2^{\infty} \frac{1}{x \ln x} dx$ take $u = \ln x$, $du = \frac{dx}{x}$

$= \int_{\ln 2}^{\ln \infty} \frac{1}{u} du = \ln x \Big|_{\ln 2}^{\infty}$

$= \underbrace{\ln \infty}_{\infty} - \ln \ln 2$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

Examples 5: $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converge/diverge

Thm: partial fractions

same as $\int_2^{\infty} \frac{2}{x^2-1} dx = \int_2^{\infty} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$

$= (\ln(x-1) - \ln(x+1)) \Big|_2^{\infty} = \ln \frac{x-1}{x+1} \Big|_2^{\infty}$

$= \underbrace{\lim_{x\to\infty} \frac{x-1}{x+1} - \ln\frac{2-1}{2+1} = \lim_{x\to\infty} \ln \frac{1-1/x}{1+1/x}}_{\ln 1 = 0} - \ln\frac{1}{3}$ $+ \ln 3$

$\therefore \sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converges (Thm obtained this via partial sums + actual sum)

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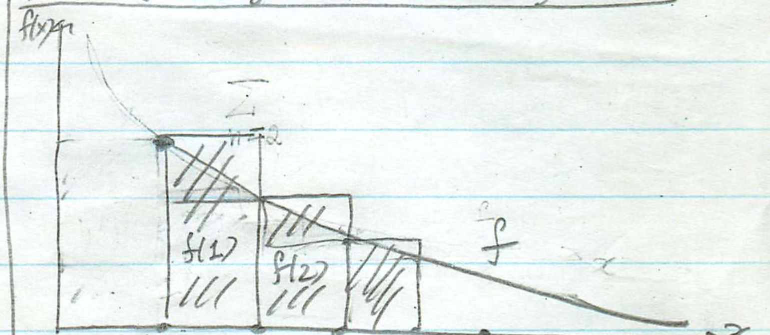
Reason for Integral and Estimating Sums



- $\sum_{n=1}^{\infty} f(n)$ converges/diverges $\Longleftrightarrow$ sequence $\{s_n\}$ of partial sums converges/diverges

$s_n = f(1) + f(2) + \dots + f(n)$

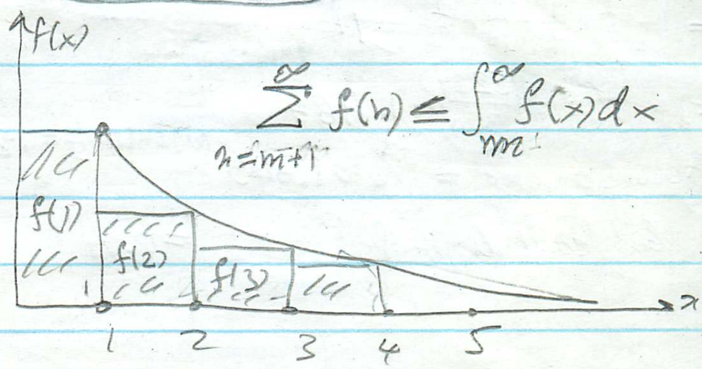
- $\int_1^{\infty} f(x) dx$ = area under the graph of f

↳ converges if area is finite

$f \geq 0$

area under the graph $\leq \sum_{n=1}^{\infty} f(n)$ b/c f decreasing

$\therefore$ if $\sum_{n=1}^{\infty} f(n)$ converges, then area is finite and $\int_1^{\infty} f(x) dx$ converges do not erase



$$\sum_{n=m+1}^{\infty} f(n) \le \int_m^{\infty} f(x)\,dx$$

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$$S_n \le f(1) + \int_1^n f(x)\,dx \le f(1) + \int_1^{\infty} f(x)\,dx$$

$\uparrow$ f decreasing $\qquad\qquad \uparrow$ $f \ge 0$

if $\int_1^{\infty} f(x)dx$ converges, $\{S_n\}$ is bounded above

$S_{n+1} = S_n + f(n) \ge S_n \Rightarrow$ S_n increasing

$\therefore$ $\{S_n\}$ increasing bounded above if $\int_1^{\infty} f(x)dx < \infty$

$\Rightarrow$ converges, and so does $\sum_{n=1}^{\infty} f(n)$

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do not

Estimating convergent $\sum_{n=1}^{\infty} f(n)$

if f positive, decreasing (continuous)

$$0 \le \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{n=m} f(n+1) = \sum_{n=m+1}^{\infty} f(n) \le \int_m^{\infty} f(x)\,dx$$

$\uparrow$ usually $<$

Example 6: Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ within $\frac{1}{5}$

(1) Find m s.t. $\int_m^{\infty} \frac{1}{x^2}dx \le \frac{1}{5}$

$\Rightarrow \sum_{n=m+1}^{\infty} \frac{1}{n^2} < \frac{1}{5}$

(2) Compute $\sum_{n=1}^{n=m}$

$$\int_m^{\infty} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_m^{\infty} = \frac{1}{m} \le \frac{1}{5}$$

$\Rightarrow$ $m=5$ will do

over $\Rightarrow$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{n=5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = 1 + \frac{900 + 400 + 225 + 144}{9 \cdot 16 \cdot 25}$$

$$= 1 + \frac{1669}{3600} = \frac{5269}{3600} \approx 1.464$$

Actual answer $= \pi^2/6 \approx 1.645$ within $\frac{1}{5} = .2$

but can't drop-off $\frac{1}{5^2} = \frac{1}{25} = 0.04$