MAT 127

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03/22/22

Series = sum of terms in a sequence

sequence $\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots \to$ converge/div

series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ converge/div

partial sums: $s_n = a_1 + a_2 + \ldots + a_n$ conv/div

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ (by dfn)

$\sum_{n=1}^{\infty} a_n$ converges/diverges if $\{s_n\}$ does (by dfn)

Geometric Sequence and Series

sequence $\{cr^n\}_{n\geq 0} = c, cr, cr^2, cr^3, \ldots$

series $\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \ldots$

partial sums $s_n = c + cr + \ldots + cr^n = \begin{cases} c(n+1) & \text{if } r=1 \\ \frac{c}{1-r}(1-r^{n+1}) & \text{if } r\neq 1 \end{cases}$

$\{s_n\}$ converges if $|r|<1$, to $\frac{c}{1-r}$

diverges if $|r|\geq 1$ and $c\neq 0$

If $c\neq 0$, geometric sequence $c, cr, cr^2, \ldots$

- converges to 0 if $|r|<1$, c if $r=1$
- diverges if $|r|>1$ or $r=-1$

If $c\neq 0$, geometric series $\sum_{n=0}^{\infty} cr^n$

- converges to $\frac{c}{1-r}$ if $|r|<1$
- diverges if $|r|\geq 1$

Example 1: Write $2.2\overline{36} = 2.2363636\ldots$ as single fraction p/q, p, q = integers

$$2.2\overline{36} = 2.2 + \underbrace{.036 + .036\tfrac{1}{100} + .036\tfrac{1}{100^2} + \ldots}_{\text{geometric series with } c = .036,\ r=\frac{1}{100}}$$

$|r|<1 \Rightarrow$ converges to $\frac{c}{1-r} = \frac{.036}{1-\frac{1}{100}}$

$$2.2\overline{36} = \frac{22}{10} + \frac{36/1000}{1-\frac{1}{100}} = \frac{11}{5} + \frac{36/1000}{99/100}$$

$$= \frac{11}{5} + \frac{36}{99\cdot 10} = \frac{11}{5} + \frac{4}{11\cdot 10} = \frac{11}{5} + \frac{2}{11\cdot 5} = \frac{11\cdot 11+2}{55} =$$

$$= \frac{121+2}{55} = \boxed{\frac{123}{55}}$$

Example 2: $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ Converges or diverges? If converges, find sum.

Partial fractions:

$$\frac{1}{(n-1)(n+1)} = \frac{1}{-(\ ?\ )}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$= \frac{1}{+1-(-1)}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

Converges or diverges? Look at partial sums

$$s_n = \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) =$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \ldots \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} + \frac{1}{2} - \frac{1}{(n-1)+1} - \frac{1}{n+1}$$

$\therefore s_n = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$ converge/diverges?

Converges, $\lim_{n\to\infty} s_n = \frac{3}{2}$

$\Rightarrow \sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converges, $= \lim_{n\to\infty} s_n = \frac{3}{2}$

Less formally:

$$\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \ldots$$

$$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

This is risky!

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Example 3: $\sum_{n=1}^{\infty} (\cos 1/n - \cos 1/(n+1))$

converge/diverge? find sum if converges

Look at partial sums

$s_n = \sum_{k=1}^{n} (\cos 1/k - \cos 1/(k+1))$

$= (\cos 1/1 - \cancel{\cos 1/2}) + (\cancel{\cos 1/2} - \cancel{\cos 1/3}) + \dots + (\cos 1/n - \cos 1/(n+1))$

$= \cos 1 - \cos\left(\frac{1}{n+1}\right)$

$s_n = \cos 1 - \cos \frac{1}{n+1}$ converges or diverges

HW: s_n converges to $\cos 1 - \cos 0 = \cos 1 - 1$

$\Rightarrow \sum_{n=1}^{\infty} (\cos 1/n - \cos 1/(n+1))$ converges

$= \lim_{n\to\infty} s_n = \boxed{\cos 1 - 1}$

Informal way:

$\sum_{n=1}^{\infty} (\cos 1/n - \cos 1/(n+1)) =$

$= (\cos 1/1 - \cancel{\cos 1/2}) + (\cancel{\cos 1/2} - \cancel{\cos 1/3}) + (\cancel{\cos 1/3} - \cancel{\cos 1/4}) + \dots$

$= \cos 1$ wrong!

because $\cos 1/n \not\to 0$ as $n \to \infty$

$\therefore \sum_{n=1}^{\infty} (b_n - b_{n+m})$ converges if $\{b_n\}$ converges

some sequence, e.g. $\cos(1/n)$

m = some integer >0, e.g. $m=1$

telescoping cancellation ("informal way") works to get the sum if $\lim_{n\to\infty} b_n = 0$

E.g. $\sum \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$ works, $\sum (\cos \frac{1}{n} - \cos \frac{1}{n+1})$ does not work

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Quick ways to tell converge/diverge?

Most important divergence test for series:

1) if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

2) if $\{a_n\}$ does not converge or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Warning: if $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ may or may not converge!

Example 4: $\sum_{n=1}^{\infty} (\ln n - \ln(n+1))$ converges or diverges?

harmonic example? $1 + \frac{1}{2} > 1$, $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$ ⋮

$\lim_{n\to\infty} (\ln n - \ln(n+1)) = \lim_{n\to\infty} \ln \frac{n}{n+1}$

$= \lim_{n\to\infty} \ln\left(\frac{1}{1+1/n}\right) = \ln 1 = 0$

$\Rightarrow$ cannot tell from "most important test"

$\Rightarrow$ look at partial sums

$s_n = \sum_{k=1}^{n} (\ln k - \ln(k+1))$

$= (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \dots + (\ln n - \ln(n+1))$

$= \ln 1 - \ln(n+1) = -\ln(n+1)$

$\therefore s_n = -\ln(n+1)$ converges or diverges?

$\Rightarrow \sum_{n=1}^{\infty} (\ln n - \ln(n+1))$ diverges

even though $\lim_{n\to\infty} (\ln n - \ln(n+1)) = 0$

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Example 5: geometric series $\sum_{n=0}^{\infty} cr^n$, $c \neq 0$

diverges if $|r| \geq 1$ because

cr^n diverges if $|r| > 1$ or $r = -1$

$\lim_{n\to\infty} cr^n = c \neq 0$ if $r = 1$

Reason for most important divergence test:

(1) $\Leftrightarrow$ (2); $\therefore$ enough to consider (1)

look at partial sums:

$S_n = a_1 + \dots + a_n$

$\Rightarrow a_n = S_n - S_{n-1}$ if $n \geq 2$

if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} S_n = S$ for some S

$\Rightarrow S_n - S \to 0$ as $n \to \infty$

$\Rightarrow a_n = S_n - S_{n-1} = (S_n - S) - (S_{n-1} - S) \to 0 - 0 = 0$ as $n \to \infty$