MAT 127

03/10/22

Last time: Sequences, converge/diverge

$a_1, a_2, a_3, \ldots$

either approaches some number a

"converges to a", $\lim_{n\to\infty} a_n = a$

or no such number exist, $\{a_n\}$ diverges

"two limits" = no limit

Tu: used 2nd case for $a_n = \frac{7^n}{n!} \geq 0$ (n-9)

$a_{n+1} = \frac{7}{n+1} \cdot a_n \leq a_n$ if $n \geq 6$

decreasing, bounded below $\to$ converges

to find limit, take $\lim_{n\to\infty}$ of $a_{n+1} = \frac{7}{n+1} a_n$

Trick: suppose converges and $\lim_{n\to\infty} a_n = a$.

Find a first, then apply Convergence Test 1

Take lim of $a_{n+1} = \sqrt{20+a_n}$:

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{20+a_n} = \sqrt{20 + \lim_{n\to\infty} a_n} = \sqrt{20+a}$

$= \lim_{n\to\infty} a_n = a$

$\therefore$ $a = \sqrt{20+a}$ solve this for a!

Show: a_n bounded above (≤ 5), increasing

$\Rightarrow$ converges by Monotonic Convergence Thm

Claim 1: $a_n \leq 5$ Check by induction.

$a_1 = \sqrt{20} \leq 5$ ✓

if $a_n \leq 5$, then $a_{n+1} = \sqrt{20+a_n} \leq \sqrt{20+5} \leq 5$ ✓

Claim 2: $\{a_n\}$ increasing $\Leftrightarrow a_{n+1} = \sqrt{20+a_n} \geq a_n$

Convergence Test 1: Monotone Convergence Thm

$$\boxed{\begin{array}{l} \text{if } \underbrace{a_n \leq a_{n+1} \leq a_{n+2} \leq \ldots}_{\text{increasing}} \text{ and } \underbrace{\leq M}_{\text{bounded above,}} \\ \text{or } \underbrace{a_n \geq a_{n+1} \geq a_{n+2} \geq \ldots}_{\text{decreasing}} \text{ and } \underbrace{\geq m}_{\text{bounded below}} \\ \text{then } \{a_n\} \text{ converges} \end{array}}$$

Another example: $\sqrt{20}, \sqrt{20+\sqrt{20}}, \sqrt{20+\sqrt{20+\sqrt{20}}}$,

$\sqrt{20+\sqrt{20+\sqrt{20+\sqrt{20}}}}, \ldots$

$\Rightarrow a_1 = \sqrt{20}$, $a_{n+1} = \sqrt{20+a_n}$

Converges or diverges?

if converges, $\lim a_n = ?$

$a^2 = 20 + a \rightsquigarrow \underbrace{a^2 - a - 20 = 0}$

$(a-5)(a+4) = 0 \Rightarrow a = 5, -4$

$-4 \neq \sqrt{20-4} \Rightarrow a = 5$

$\Rightarrow \lim_{n\to\infty} a_n = 5$ if the limit exist

Still need to check $\{a_n\}$ convergence

do not erase

$20 + a_n \geq a_n^2 \Leftrightarrow \underbrace{a_n^2 - a_n - 20 \leq 0}$

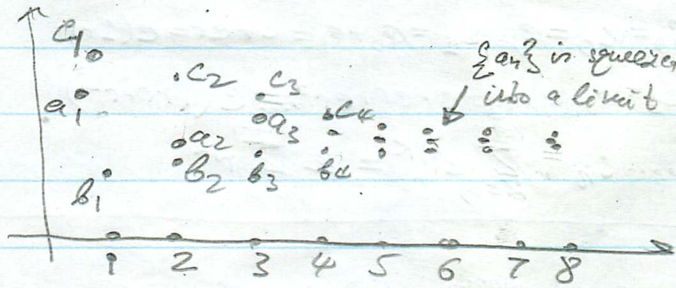
$(a_n-5)(a_n+4) \leq 0$ b/c $a_n \leq 5$ ✓

$\therefore$ $\{a_n\}$ increasing, bounded above $\Rightarrow$ converges

Convergence Test 2: Squeeze Theorem for Sequences

> If $\{a_n\}, \{b_n\}, \{c_n\}$ are such that
> $b_n \le a_n \le c_n$ for all n ($\ge$ some N),
> $\{b_n\}, \{c_n\}$ converge, and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$
> then $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$

Graph way to see this:



Example 2: $a_n = \frac{7^n}{n!}$, $n \ge 1$.

Take $b_n = 0$ for all n, $b_n \le a_n$, $\lim\limits_{n\to\infty} b_n = 0$ ✓

$a_{7+n} = \frac{7^{7+n}}{(7+n)!} = \frac{7^7}{7!} \cdot \frac{7}{8} \cdots \frac{7}{7+n} \le a_7 \cdot \left(\frac{7}{8}\right)^n$

Take $c_{7+n} = a_7 \cdot \left(\frac{7}{8}\right)^n$ geometric sequence, $\lim\limits_{n\to\infty} c_n = 0$

$a_n \le c_n$ for all $n \ge 7$

∴ $b_n \le a_n \le c_n$

$\overset{\shortparallel}{0} \to 0$ as $n\to\infty$ $\qquad$ $\overset{\shortparallel}{a_7 \cdot \left(\frac{7}{8}\right)^n} \to 0$ as $n\to\infty$

Squeeze Thm for Sequences $\Longrightarrow$ $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n = 0$

∴ Sequence = infinite string of numbers

could converge to $\underbrace{\text{some number}}_{\lim\limits_{n\to\infty} a_n}$ or diverge

only one limit if it exists

2 Convergence Tests for Sequences:

(1) Monotonic Sequence Thm (2) Squeeze Thm for Sequences (3) L'Hospital's

Series = sum of terms in a sequence

Sequence $\{a_n\} = a_1, a_2, a_3, \dots$

Series $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

→ Infinite sum; may exist or not

Series converges $\qquad$ diverges

Meaning of $\sum\limits_{n=1}^{\infty} a_n$

= limit of sequence of partial sums

$s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, … $s_n = \sum\limits_{k=1}^{n} a_k$, …

if the limit exists ← $\lim\limits_{n\to\infty} s_n$, not $\lim\limits_{n\to\infty} a_n$

series $\sum\limits_{n=1}^{\infty} a_n$ converges

if $\lim\limits_{n\to\infty} s_n$ does not exist, $\sum\limits_{n=1}^{\infty} a_n$ diverges

Example 3: (From geometric sequence to series)

> Thm if $c \ne 0$, geometric sequence $c, cr, cr^2, cr^3, \dots$
> converges if $-1 < r \le 1$ (to 0 if $|r| < 1$, to c if $r = 1$)
> diverges if $r \le -1$ or $r > 1$

over $\Longrightarrow$

geometric series: $\sum_{n=0}^{\infty} cr^n$

converges or diverges?

must look at sequence of partial sums

$s_0 = a_0 = c, \quad s_1 = a_0 + a_1 = c + cr = c(1+r)$

$s_2 = a_0 + a_1 + a_2 = c + cr + cr^2 = c(1+r+r^2)$

$\dots\dots s_n = \sum_{k=0}^{k=n} cr^k = c \sum_{k=0}^{k=n} r^k$

Recall: $1 - r^{n+1} = (1-r)\underbrace{(1+r+r^2+\dots\dots+r^n)}_{(1-r)+(r-r^2)+(r^2-r^3)+\dots+(r^n-r^{n+1})}$

$\Rightarrow \frac{1-r^{n+1}}{1-r} = \sum_{k=0}^{n} r^k$

$\Rightarrow s_n = c\sum_{k=0}^{n} r^k = \frac{c}{1-r}(1-r^{n+1})$

Does s_n converge or diverge if $c \neq 0$?

do not erase

$\sum_{n=0}^{\infty} cr^n$ converges $\Leftrightarrow \{s_n\}$ converges

$\hookrightarrow$ not $\{a_n = cr^n\}$!

Does s_n converge?

$r=1$: $s_n = c\sum_{k=0}^{n} 1^n = c\cdot(n+1)$

does not converge if $c \neq 0$

$\rightsquigarrow$ Assume $r \neq 1$

do not erase

$s_n = \frac{c}{1-r}(1-r^{n+1}), \quad c \neq 0, r \neq 1$

if $|r|<1$, $r^{n+1} \to 0$

$\Rightarrow \{s_n\}$ converges, $\sum_{n=0}^{\infty} cr^n = \lim_{n\to\infty} s_n = \frac{c}{1-r}$

if $|r|>1$, $|r|^{n+1} \to \infty \Rightarrow s_n$ diverges

if $r=-1$ r^{n+1} jumps between -1 and 1

$\Longrightarrow \{s_n\}$ diverges

$\boxed{\text{If } c \neq 0 \text{, the geometric series } \sum_{n=0}^{\infty} cr^n \text{ converges if } |r|<1 \ (\text{to } \tfrac{c}{1-r}) \text{, diverges if } |r| \geq 1}$

geometric series converges $\Rightarrow$ geometric seq. converges

but for $r=1$: geometric sequences converges, series diverges

$\sum a_n$ converges $\Rightarrow \lim a_n = 0$