Part II: Sequences, Series, and Power Series

sequence = infinite string of numbers

$a_1, a_2, a_3, a_4, \ldots$

could be written out or

given by a formula $a_n = f(n)$, $n \geq 1$ (or ≥ 0)

or described in some way

confusing point: convergence tests for sequences vs series

Example 1: $1, -1, 1, -1, 1, -1, \ldots$

description: $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$

formula: $a_n = -(-1)^n = (-1)^{n+1}$

Example 2: $a_n = \dfrac{(-2)^n}{n^2}$ $\quad n \geq 1$

list: $a_1 = \dfrac{(-2)^1}{1^2} = -2 \quad a_2 = \dfrac{(-2)^2}{2^2} = 1$

$a_3 = \dfrac{(-2)^3}{3^2} = -\dfrac{8}{9} \quad a_4 = \dfrac{(-2)^4}{4^2} = 1$

$a_5 = \dfrac{(-2)^5}{5^2} = -\dfrac{32}{25} \quad a_6 = \dfrac{(-2)^6}{6^2} = \dfrac{64}{36} = \dfrac{16}{9}$

list: $-2, 1, -\dfrac{8}{9}, 1, -\dfrac{32}{25}, \dfrac{16}{9}, \ldots$

Example 3: $-\dfrac{1}{2}, \dfrac{\sqrt{2}}{3}, -\dfrac{\sqrt{3}}{4}, \dfrac{2}{5}, -\dfrac{\sqrt{5}}{6}, \dfrac{\sqrt{6}}{7}, \ldots$

formula? $a_n = ?$

Signs alternate $= \begin{cases} -1 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases} = (-1)^n$

numerator $= \sqrt{n}$

denominator $= n+1$

$\therefore \boxed{a_n = (-1)^n \dfrac{\sqrt{n}}{n+1}, \ n \geq 1}$

do not erase

Example 4: a_n = n-th decimal digit of

$\pi \approx 3.1415927\ldots$

$\rightarrow 1, 4, 1, 5, 9, 2, 7$

formula for $a_n = ?$

"no such thing"

Example 5 (Fibonacci numbers):

recursive definition: $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ if $n \geq 0$

list: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

formula? a_n = something in terms of n?

Problem G (not hard)

Important property of a sequence:

Does it converge to some number or not (diverge)?

If so what number?

$\hookrightarrow$ limit of the sequence, $\lim\limits_{n\to\infty} a_n = ?$

"diverges" = "no limit"

Example 1': $a_n = -(-1)^n \qquad n \ge 1$

$1, -1, 1, -1, 1, -1, \ldots$

Approaches any number?

No: Keeps on jumping between -1 and 1

$$\underset{a_2, a_4, a_6, \ldots}{-1} \qquad \underset{a_1, a_3, a_5}{1}$$

$\therefore$ $\boxed{\text{diverges}}$

Example 3': $a_n = (-1)^n \dfrac{\sqrt{n}}{n+1} \qquad n \ge 1$

Approaches any number?

$$a_n = (-1)^n \frac{\sqrt{n}/\sqrt{n}}{n/\sqrt{n} + 1/\sqrt{n}} = \underbrace{(-1)^n}_{\pm} \frac{1}{\sqrt{n} + 1/\sqrt{n}} \xrightarrow{\text{as } n \to \infty} 0$$

$\therefore \lim_{n \to \infty} a_n = 0$, sequence $\boxed{\text{converges}}$

Example 3'': $a_n = (-1)^n \dfrac{\sqrt{n^2+1}}{n+1} \qquad n \ge 1$

Converges or diverges?

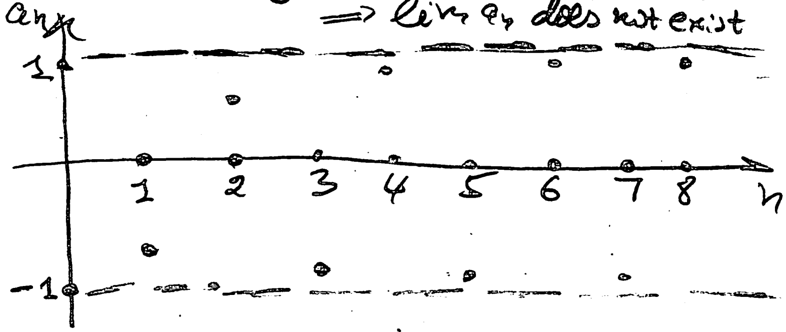
$$a_n = (-1)^n \frac{\sqrt{n^2+1}/n}{n/n + 1/n} = \underbrace{(-1)^n}_{\pm} \frac{\sqrt{1 + 1/n^2}}{1 + 1/n} \xrightarrow{\text{as } n \to \infty} 1$$

$\therefore$ a_n keeps on jumping between close to 1 and (-1)

$\Longrightarrow$ does not converge $\boxed{\text{diverges}}$

"Graph way" $a_{odd} \to -1$, $a_{even} \to 1$

$\Rightarrow \lim a_n$ does not exist



Example 5: (geometric sequence, not series yet)

$c, cr, cr^2, cr^3, \ldots$ $(c \neq 0)$

Is this convergent?

$|r|<1 \Rightarrow r^n \to 0 \Rightarrow cr^n \to 0$ as $n \to \infty$

$|r|>1 \Rightarrow |r|^n \to \infty \Rightarrow cr^n$ diverge (if $c \neq 0$)

e.g. $3, 9, 27, 81, 243, \ldots$ $r=3$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$ $r=\frac{1}{3}$

$r=1$: $c, c, c, \ldots$ converges to c.

$r=-1$: $c, -c, c, -c$ alternates between c and $-c$ $\Rightarrow$ diverges

$\therefore$ The geometric sequence $c, cr, cr^2, cr^3, \ldots$

- converges if $-1 < r \le 1$ (to 0 if $-1<r<1$)
- diverges if $r \le -1$ or $r > 1$

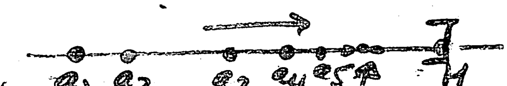
Note: statement for geometric series a bit different

Convergence test 1: Monotonic Sequence Thm

(a) if $a_n \le a_{n+1}$ and $a_n \le M$ for all n ($\ge$ some N), then $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n \le M$

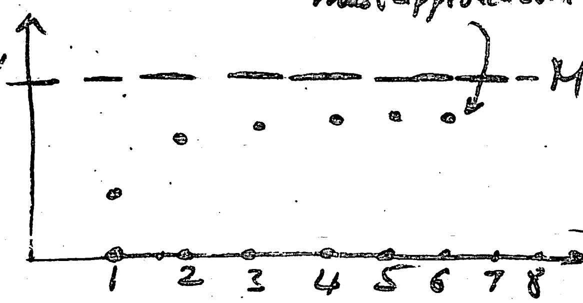
(b) if $a_n \ge a_{n+1}$ and $a_n \ge m$ for all n ($\ge$ some N) then $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n \ge m$

Why (a)?



must approach something

Graph way



Example 6: $a_n = \frac{7^n}{n!}$, $n \ge 1$, converges or diverges?

Use (b): $a_n \ge 0$

$a_{n+1} = \frac{7^{n+1}}{(n+1)!} = \frac{7^n}{n!} \cdot \frac{7}{n+1} = \frac{7}{n+1} a_n \le a_n$ if $n \ge \overset{N}{6}$

$\therefore$ a_n $\boxed{\text{converges}}$.

$\lim\limits_{n\to\infty} a_n = a = ?$

Trick: $a_{n+1} = \frac{7}{n+1} a_n$

Take $\lim\limits_{n\to\infty}$ of both sides:

$$\underbrace{\lim_{n\to\infty} a_{n+1}}_{\lim\limits_{n\to\infty} a_n = a} = \lim_{n\to\infty} \frac{7}{n+1} a_n = \underbrace{\left(\lim_{n\to\infty} \frac{7}{n+1}\right)}_{0} \cdot \underbrace{\lim_{n\to\infty} a_n}_{a}$$

$\therefore \lim\limits_{n\to\infty} \frac{7^n}{n!} = 0$

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do not erase

$a_n \le 10$

Example 7: $\sqrt{20}, \sqrt{20+\sqrt{20}}, \sqrt{20+\sqrt{20+\sqrt{20}}}, \ldots$

converges or diverges?

recursive re-definition: $a_1 = \sqrt{20}$

$a_{n+1} = \sqrt{20 + a_n}$, $n \ge 1$

Trick: if the limit exists, what is it?

if $\lim\limits_{n\to\infty} a_n = a$, $a = ?$

over $\Rightarrow$

Previous trick: $a_{n+1}=\sqrt{20+a_n}$; take $\lim\limits_{n\to\infty}$

$$\underbrace{\lim_{n\to\infty} a_{n+1}}_{\lim\limits_{n\to\infty} a_n = a} = \lim_{n\to\infty}\sqrt{20+a_n} = \sqrt{20+\underbrace{\lim_{n\to\infty} a_n}_{a}}$$

$\therefore\ a=\sqrt{20+a} \Rightarrow a^2=20+a$

$a^2-a-20=0 \sim (a-5)(a+4)=0$

$\Rightarrow a=5$ or $\cancel{a=-4}$ $(\neq\sqrt{20+a})$

$\therefore\ \lim\limits_{n\to\infty} a_n=5$ <u>if</u> $\{a_n\}$ converges

<u>Use (a)</u> <u>Claim 1:</u> $a_n\le 5$. True for $n=1$

if $a_n\le 5$, then $a_{n+1}=\sqrt{20+a_n}\le\sqrt{20+5}=5$ ✓

<u>Claim 2:</u> $a_n \overset{?}{\le} a_{n+1}$

$a_{n+1}=\sqrt{20+a_n}\overset{?}{\ge}a_n\ge 0 \iff 20+a_n \overset{?}{\ge} a_n^2$

$a_n^2-a_n-20\overset{?}{\le}0$, $(a_n-5)(a_n+4)\overset{?}{\le}0$ yes if $a_n<5$ ✓