MAT 127

3/3/22

Continue: Systems of 2 autonomous 1st-order differential equations

and two-species interactions

1-species population ⟷ 1 autonomous eqn

2 populations ⟷ 2 intertwined

Main Example: $R(t)$ = # rabbits at time t

$W(t)$ = # wolves at t

$$\begin{cases} \frac{dR}{dt} = \frac{1}{20}R - \frac{1}{500}RW \\ \frac{dW}{dt} = -\frac{1}{10}W + \frac{1}{1000}RW \end{cases} \quad (R,W) = (R(t), W(t))$$

$W=0$: $\frac{dR}{dt} = \frac{1}{20}R$ exponential growth eqn.

⇒ # of rabbits grows exponentially if no wolves

$$\frac{dR}{dt} = \frac{1}{20}R - \frac{1}{500}RW$$

wolves have negative effect on rabbits ✓

$R=0$: $\frac{dW}{dt} = -\frac{1}{10}W$ exponential decay eqn.

⇒ # wolves decays to 0 if no rabbits

$$\frac{dW}{dt} = -\frac{1}{10}W + \frac{1}{1000}RW$$

rabbits have positive effect on wolves

∴ reasonable model for rabbits-wolves interaction

About solutions $(R,W) = (R(t), W(t))$ of the system

(1) What are the constant/equilibrium solutions?

$(R,W) = (\text{const}, \text{const}) \Rightarrow \frac{dR}{dt} = 0, \ \frac{dW}{dt} = 0$

$$\Rightarrow \begin{cases} \frac{1}{20}R(1-\frac{1}{25}W) = 0 \\ -\frac{1}{10}W(1-\frac{1}{100}R) = 0 \end{cases} \Rightarrow \begin{cases} R=0 \text{ or } W=25 \\ W=0 \text{ or } R=100 \end{cases}$$

Constant solutions: $(R(t), W(t)) = (0,0)$ for all t

$(R(t), W(t)) = (100, 25)$

CAUTION: must choose 1 condition from each line of

$$\begin{cases} R=0 \text{ or } W=25 \\ W=0 \text{ or } R=100 \end{cases}$$

E.g. 1st line $R=0$ ⟶ $W=0$ / ~~$R=100$~~ 2nd line

$(W,R) = (25,0)$ not a constant solution

must pick something from 2nd line

Significance of constant solutions:

$(R,W) = (0,0)$ no rabbits, no wolves

$(R,W) = (100,25)$ 100 rabbits precisely enough to support 25 wolves and be contained by them

Two ways to "graph solutions" $(R,W)=(R(t),W(t))$

(1) curve traced by $(R(t),W(t))$ as $t\to\infty$ in RW-plane

→ phase trajectory

(2) graphs of $R=R(t)$ and $W=W(t)$ as functions of t

with same t, but different R-axis, W-axis

Goal: Sketch one from the other

Thu: did this for $(x(t),y(t))=(2\cos t, \sin t)$

(do not erase)

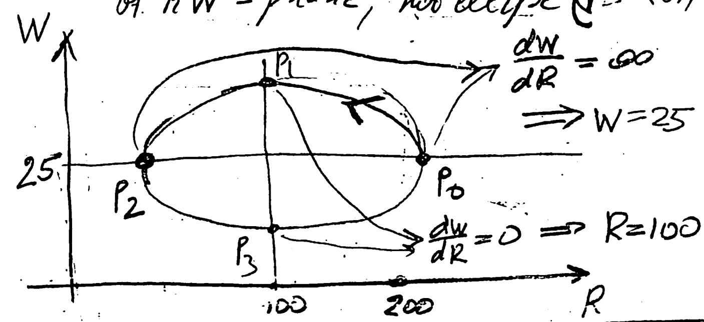
$(R,W)=(R(t),W(t))$ solution of the system traces some curve in RW-plane

Slope of curve at (R,W):

$$\frac{dW}{dR}=\frac{dW/dt}{dR/dt}=\frac{-\frac{1}{10}W+\frac{1}{1000}RW}{\frac{1}{20}R-\frac{1}{500}RW}=-\frac{W(100-R)}{2R(25-W)}$$

separable equation ⇒ can solve (Problem F-(a))

get simple closed curves in 1st-quadrant of RW-plane, not ellipse (F-(b))



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extremal left/right points on phase trajectory

$\Longleftrightarrow \frac{dW}{dR}=\infty \Longleftrightarrow \frac{dR}{dt}=0$ ($\Leftrightarrow W=25$ in this case)

extremal up/down points

$\Longleftrightarrow \frac{dW}{dR}=0 \Longleftrightarrow \frac{dW}{dt}=0$ ($\Leftrightarrow R=100$ in this case)

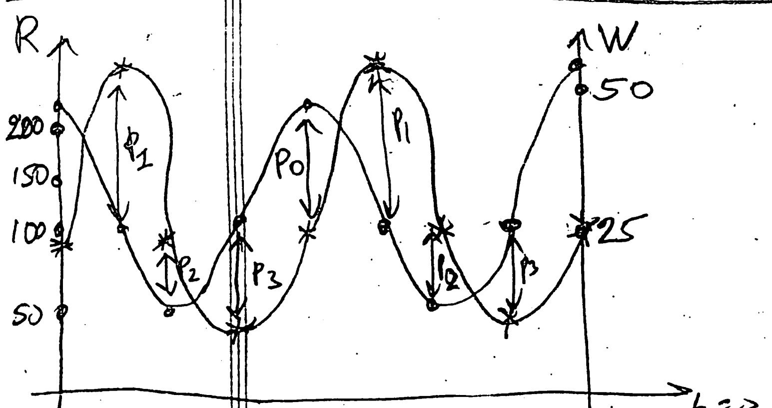
Sketch graphs of $R=R(t)$ and $W=W(t)$ as function of time from phase trajectory

Key points on trajectory = turns in direction

$P_0=(220,25)$ $P_1=(100,52)$

$P_2=(50,25)$ $P_3=(100,12)$

then repeat; W-range = 50-220; R-range = 12-52



R: maxs/mins ⇔ W=25; W: max/mins ⇔ R=100

(1) Mark "extremal points"

(2) Connect R-points by smoothing going up or down only between nearby points

Same for W-points

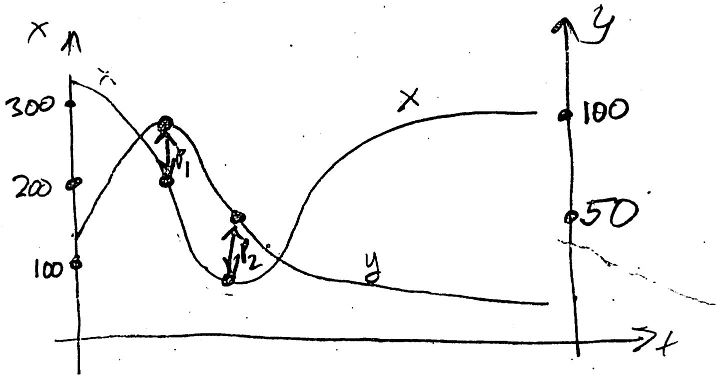
Caution: (1) Keep R and W points separate

(2) R-axis and W-axis may have different scales

Color chalk helps

over ⇒

From graphs to phase trajectory



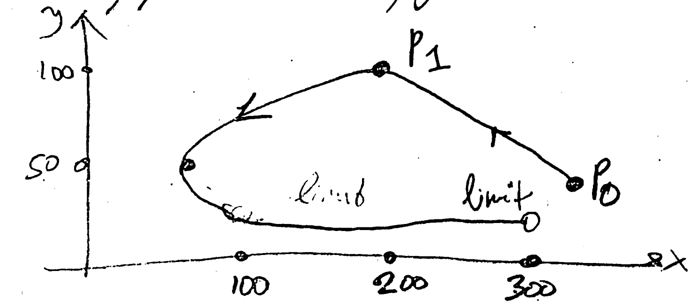
Important Points

Starting: $x=320$, $y=48 \rightarrow P_0 = (320, 48)$

1st y-~~peak~~: $x = 200$, $y = 100 \rightarrow P_1 = (200, 100)$ → extremal top point

1st x-min: $x = 80$, $y = 50 \rightarrow P_2 = (80, 50)$ → extremal left point

Next asymptotes: $x \rightarrow 100$, $y \rightarrow 20$



(1) mark extremal points by dots but not the limit (just ↗)

(2) Connect all by smooth curve in correct order (increasing t) <u>without</u> changing general direction between extremal points

general direction = left & up, left & down, ...