Remaining topic in Part I:

Systems of 2 autonomous 1-st order differential eqn.

t = independent variable, "time"

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \qquad (x,y) = (x(t), y(t))$$

must find/specify (x,y) at the same time

Example 1: $\begin{cases} \dfrac{dx}{dt} = -2y \\ \dfrac{dy}{dt} = \dfrac{1}{2}x \end{cases}$ $\quad (x,y) = (x(t), y(t))$

Claim: $(x(t), y(t)) = (2\cos t, \sin t)$ is a solution of the system

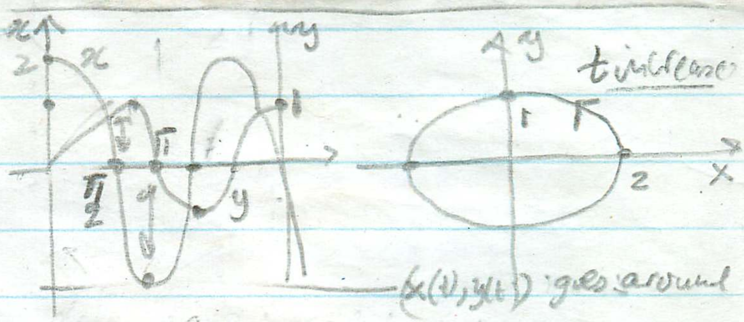
Plug in to check: $\dfrac{dx}{dt} = -2\sin t = -2y$ ✓
$\dfrac{dy}{dt} = \cos t = \dfrac{1}{2}x$ ✓

Important: $(x(t), y(t)) = (2\cos t, \sin t)$ is a solution, not $x(t) = 2\cos t$ and $y(t) = \sin t$ separately

Two ways to "sketch solutions"

(1) graphs $x = x(t)$, $y = y(t)$ as functions of t

(2) curve traced by $(x(t), y(t))$ in xy-plane
↳ phase trajectory: $\left(\dfrac{x(t)}{2}\right)^2 + y(t)^2 = 1 \Rightarrow$ ellipse



$(x(t), y(t))$ goes around as t increases

Usually can't solve systems, but can

(1) say something (e.g. find constant solutions)

$$\begin{cases} \dfrac{dx}{dt} = -2y = 0 \\ \dfrac{dy}{dt} = \dfrac{1}{2}x = 0 \end{cases} \Rightarrow (x(t), y(t)) = (0,0) \text{ is constant solution}$$

(2) read off phase trajectory from graphs and graphs from phase trajectory (without t-scale)

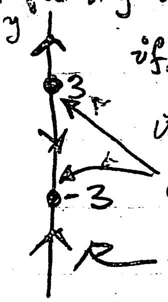
1-equation case: autonomous 1st-order differential eqn.

Example 2: $y' = y^2 - 9$, $\quad y = y(t)$

(1) constant solutions: $y' = y^2 - 9 = 0 \Rightarrow y(t) = 3,\ y(t) = -3$

(2) "phase trajectory" = paths taken by $y(t)$ on y-axis as t increases

Phase trajectory for $y'=(y-3)(y+3)$ $\quad y=y(t)$



if $y(0)=3$ or -3, $y(t)=3$ or -3 for all

if $y(t)$ starts at ± 3, it stays there

equilibrium/stationary points

if $y(t)>3$ for some t, $y'(t)>0 \Rightarrow y(t)$ increases

$y(t)$ flows up

if $y(t)<-3$, $y(t)$ also flows up

if $-3<y(t)<3$, $y(t)$ flows down

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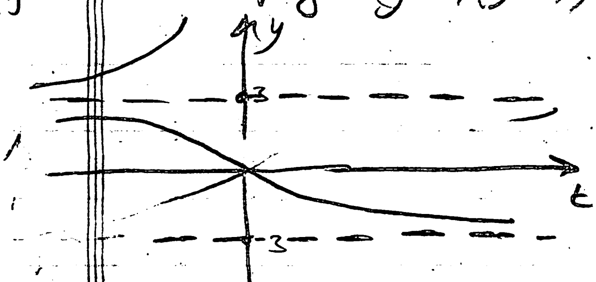
graph of a solution of $y'=(y-3)(y+3)$, $y=y(t)$



can get from phase trajectory on y-axis

Would like to do similar things for systems of 2 autonomous 1st-order differential eqn.

$$\begin{cases} \dfrac{dx}{dt}=f(x,y) \\ \dfrac{dy}{dt}=g(x,y) \end{cases} \qquad (x,y)=(x(t),y(t))$$

(1) find equilibrium points ← easy

(2) sketch phase trajectories in xy-plane of graph