~~⇒ HW3 due~~
HW4 due Th by 2:20 p.m.
Tu, 9/29: Mon...
W, 9/30 OHs: 9-...

MAT 127

~~09/24/09~~
02/17/22

Board #3

Exponential Growth/Decay Equation

The solution to the initial-value problem

$y' = ry$, $y = y(t)$, $y(0) = y_0$, (r const)

is $y(t) = y_0 e^{rt}$

In problems: start with correct equation, then figure out the constants: y_0, r, K (only in logistic eqn.)

Tu: did this in 2 examples for exponential eqn.
Works similarly for logistic equation

Can do this via direction field

$y' = ry\left(1 - \frac{y}{K}\right)$, $y = y(t)$ $r, K > 0$

0-slopes on lines $y = 0$, $y = K$

+ slopes if $0 < y < K$, $\to 0$ as $y \to 0$ or K
independent of t ⟺ under horizontal shifts

− slopes if $y < 0$ or $y > K$, $\to 0$ as $y \to 0$ or K
independent of t ⟺ under horizontal shifts

Can sketch solution curves: direction field does not change under horizontal shifts

⇒ horizontal shift of a solution curve is another solution curve

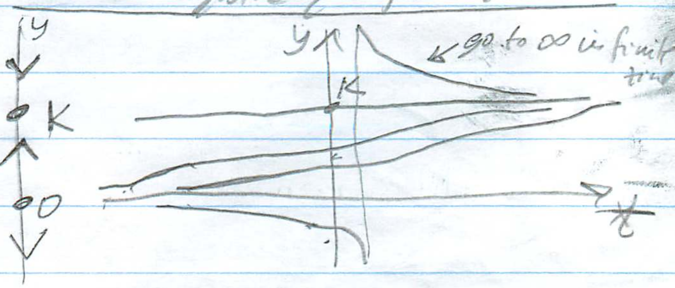
Logistic Growth Equation

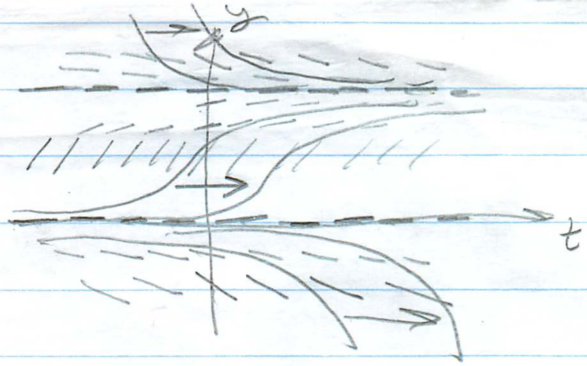
The solution to initial-value problem

$y' = ry\left(1 - \frac{y}{K}\right)$, $y = y(t)$, $y(0) = y_0$, is

$$y(t) = \frac{K}{1 - \frac{y_0 - K}{y_0} e^{-rt}}$$

Sketch ~~from~~ logistic eqn from solution





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2. $\boxed{\text{If } y = y(t) \text{ is a solution, so is } \tilde{y}(t) = y(t-a)}$

graph of $\tilde{y}$ = graph of y shifted by a to the right

⇒ true for every autonomous equation

$\underbrace{y' = g(y)}_{}$, $y = y(t)$

no explicit dependence on t

#2

Next topic: 2nd-order linear homogeneous diff. ODE's equations with constant coefficients:

$$Ay'' + By' + Cy = 0, \quad y = y(x)$$

2nd-order = involves y, y', y''; no y''', etc.

linear = linear in y: no y^2, yy', y'^2, etc.

linear homogeneous: no other terms, e.g. not $y'' = 5$

constant coefficients: A, B, C = const

Back to $Ay''+By'+Cy=0$, $y=y(x)$, A, B, C = const

$A=0$: $By'+Cy=0$, $y=y(x)$ 1st-order separable

⇒ assume $A \neq 0$ (⇒ can divide by A:

$$y'' + \underbrace{b}_{B/A}\,y' + \underbrace{c}_{C/A}\,y = 0 \quad y = y(x) \;)$$

How to solve this?

(a) Compute y', y'': $y'(x) = re^{rx}$, $y''(x) = r^2 e^{rx}$

Plug in into eqn: $\underbrace{2r^2e^{rx} + re^{rx} - e^{rx}}= 0$

$e^{rx}(2r^2 + r - 1) = 0 \quad (*)$

holds for all x

$(*) \iff 2r^2 + r - 1 = 0 \iff (2r-1)(r+1) = 0$

$\iff r = 1/2,\ r = -1$

Solving $Ay''+By'+Cy=0$, $y=y(x)$, A,B,C = const, $A \neq 0$

(1) Find roots r_1, r_2 of associated polynomial

$$Ar^2 + Br + C = 0$$

$\Rightarrow e^{r_1x}, e^{r_2x}$ = solutions of diff. eqn

(2) if $r_1 \neq r_2$, $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$ is the general solution

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Important property: if $y_1 = y_1(x)$, $y_2 = y_2(x)$ are solutions of (2nd-order) homogeneous linear eqn, then so is $y(x) = C_1y_1(x) + C_2y_2(x)$, C_1, C_2 = const

Check: $y'(x) = C_1y_1' + C_2y_2'$, $y'' = C_1y_1'' + C_2y_2''$

$\Rightarrow Ay'' + By' + Cy = A(C_1y_1'' + C_2y_2'') + B(C_1y_1' + C_2y_2') + y$

$= C_1(Ay_1'' + By_1' + Cy_1) + C_2(Ay_2'' + By_2' + Cy_2) = C_1 \cdot 0 + C_2 \cdot 0 = 0$ ✓

Example (HW1 WA1B) $2y'' + y' - y = 0$, $y = y(x)$

(a) Find r so that $y(x) = e^{rx}$ is a solution of this

(b) If r_1 and r_2 are the two values you found, show that $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$ is also a solution.

Important Property ⇒ (b): if $y_1(x) = e^{r_1x}$, $y_2(x) = e^{r_2x}$ are solutions, so is $y(x) = C_1y_1(x) + C_2y_2(x)$

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do not erase

∴ To find solutions of $2y'' + y' - y = 0$, find roots r_1, r_2 of $2r^2 + r - 1 = 0$; then $y_1(x) = e^{r_1x}$, $y_2(x) = e^{r_2x}$ are solutions

$\Rightarrow$ so is $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$

Easy: if r_1 is a root of $Ar^2 + Br + C = 0$, then $y(x) = e^{r_1x}$ is a solution of $Ay'' + By' + Cy = 0$

$y' = r_1e^{r_1x}$, $y'' = r_1^2e^{r_1x}$

$\Rightarrow Ay'' + By' + Cy = e^{r_1x}\underbrace{(Ar_1^2 + Br_1 + C)}_{=0} = 0$

HW4, Problem D E

$\Rightarrow$ if $r_1 \neq r_2$ and $y = y(x)$, then $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$ for some constants C_1, C_2

over ⟹

Potential issues:

(1) r_1, r_2 not real, $r_1, r_2 = \frac{p \pm i q}{2}$

$p = -B/A$, $q = -(B/A)^2 + 4(C/A)$

(2) $r_1 = r_2$

(1), (2) $\Rightarrow$ next time

Examples: today

Example 1: Find the general solution to

$y'' - 4y = 0$, $y = y(x)$

Need to find roots of associated polynomial.

Careful with coefficients! $1 \cdot y'' - 0 \cdot y' + 4 \cdot y = 0$

$\to 1 \cdot r^2 - 0 \cdot r + 4 \cdot 1 = 0$

$\therefore$ Associated polynomial is $r^2 - 4 = 0$

Roots of $r^2 - 4 = 0$: $r = \pm 2$

$\Rightarrow$ the general solution is

$$\boxed{y(x) = c_1 e^{2r} + c_2 e^{-2r}}$$

Example 2: Find the general solution to

$y'' - y' - 6 = 0$.

Same as $1 \cdot y'' - 1 \cdot y' - 6 = 0$

$\to \underbrace{1 \cdot r^2 - 1 \cdot r - 6}_{(r-3)(r+2)} = 0$

$(r-3)(r+2) = 0 \Rightarrow r_1, r_2 = 3, -2$

$\therefore y(x) = c_1 e^{3x} + c_2 e^{-2r}$ is the general solution

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IVP

initial value solution?

On WebAssign: use A and B for C_1, C_2

LeBrun