MAT 127

02/15/22

Last week: Separable equations and some applications

<u>Tu:</u> Steps 1-5 for solving separable equations

separate variables, integrate, check for constant solutions

<u>Th:</u> finding curves with specified slopes, [illegible]

Mixing problem ⇝ set up differential equation or initial-value problems and solve it

Today: more applications

(1) Exponential growth/decay equation

(2) Logistic equation

In both cases: independent variable is (usually) t = time

$y(t)$ = some function of time

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(1). Exponential growth/decay equation:

$y'(t)$ = (rate of change) is proportional to $y(t)$

$y'(t) = r\,y(t)$ for some constant r

$r \equiv \frac{y'(t)}{y(t)}$ "relative $\underset{r>0}{\text{growth}}$/$\underset{r<0}{\text{decay}}$" rate

Typical examples:

(a) Bacteria culture; (continuously compounding interest)

if population of 100 cells grows at 15 cells/min, population of 500 cells should grow at 75 cells/min

(b) Radioactive decay ($r<0$): a fixed fraction of radioactive substance stops being radioactive in a fixed interval of time

⇝ Newton's Law of Cooling

Solution of $y'(t) = r\,y(t)$:

Separable: $\frac{dy}{dt} = ry \rightsquigarrow \frac{dy}{y} = r\,dt \rightsquigarrow \int \frac{dy}{y} = \int r\,dt$

$\rightsquigarrow \ln|y| = rt + c \rightsquigarrow |y| = e^{rt+c} = e^{c}e^{rt}$

$\rightsquigarrow y(t) = \pm A e^{rt} \rightsquigarrow \boxed{y(t) = Ce^{rt}}$ general solution

Note: $y(0) = C \cdot e^{r \cdot 0} = C$

$\therefore C = y(0)$

∴ Exponential growth/decay formula:

The solution to the initial-value problem

$y' = ry$, $y = y(t)$, $y(0) = y_0$

is $y(t) = y_0 e^{rt}$

Interesting property of $y(t) = Ce^{rt}$

$y(t+\Delta t) = Ce^{r(t+\Delta t)} = Ce^{rt} \cdot e^{r\Delta t}$

$= y(t) \cdot e^{r\Delta t}$

depends on Δt, but not t

$\therefore y(t+\Delta t) = y(t) \cdot A$

∴ $y(t)$ changes by a fixed factor every Δt (mins)

Example 1: A bacteria culture contains 100 cells and is growing at 15 cells/min at $t=0$

(a) What is the number of cells at time t mins?

(b) When will this number reach 10,000?

(c) What is the growth rate at that time?

(b) When will this number reach 1,000?

(a) Exponential growth/decay formula:

$$y(t) = y(0)\,e^{rt} = 100\,e^{rt} \quad \text{(cells)}$$

What is r? $y'(t) = r\,y(t)$ for all t

Plug in $t=0$: $15\text{cells}/\text{min} = r \cdot 100\text{cells}$

$$\Rightarrow r = .15/\text{min}$$

$$\rightsquigarrow \boxed{y(t) = 100\,e^{.15t}} \quad (t = \text{in mins})$$

(b) Find t s.t. y(t) $\boxed{\frac{20}{3}\ln 10}$

(c) Find t s.t. $y(t) = 100\,e^{.15t} = 10{,}000$

$$\rightsquigarrow e^{.15t} = 100 \rightsquigarrow .15t = \ln 100 = 2\ln 10$$

$$\rightsquigarrow t = \frac{2\ln 10}{15/100} = \frac{2\ln 10}{3/20} = \boxed{\frac{40}{3}\ln 10 \text{ mins}}$$

Faster: $100 \xrightarrow{\times 10} 1{,}000 \xrightarrow{\times 10} 10{,}000.$ same time

(d) $y'(t) = r\,y(t) \rightsquigarrow (.15/\text{min}) \cdot 10{,}000\text{cells} = 15 \cdot 100\,\text{cells}/\text{min}$

$$= \boxed{1{,}500\,\text{cells}/\text{min}}$$

Example 2: A bacteria culture contains 100cells initially and 900cells 1 hour later.

(i) What is the number of cells at t hours?

Solution: $y(t) = y(0)\,e^{rt} = 100\,e^{rt}$

What is r? $y(1) = 100e^{r\cdot 1} = 900$

$$\rightarrow e^{r} = 9 \Rightarrow r = \ln 9 = \ln 3^2 = 2\ln 3$$

$$\therefore y(t) = \boxed{100\,e^{2(\ln 3)t}}$$

$$= 100 \cdot (e^{\ln 3})^{2t} = 100 \cdot (3^2)^t = \boxed{100 \cdot 9^t}$$

either answer is fine

$100\,e^{(\ln 9)t}$ is not

II. Logistic Growth Equation:

$$y'(t) = r\,y(t)\left(1 - \frac{y(t)}{K}\right), \quad y = y(t)$$

$r, K > 0$ constants, K = carrying capacity

$y' \approx ry$ if y is very small

< 0 if $y > K$ (y exceeds car

$\to 0$ as $y \to K$

Separable $\Rightarrow$ can solve:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) = -\frac{r}{K}y(y-K) \Rightarrow \frac{dy}{y(y-K)} = -\frac{r}{K}dt$$

$$\Rightarrow \int \frac{dy}{y(y-K)} = -\frac{r}{K}t + C$$

Quick PFs: $\dfrac{1}{y(y-K)} = \dfrac{1}{-0-(-K)}\left(\dfrac{1}{y-K} - \dfrac{1}{y-0}\right)$ do not cross

$$= \frac{1}{K}\left(\frac{1}{y-K} - \frac{1}{y}\right)$$

$$\therefore \frac{1}{K}\underbrace{\int\left(\frac{1}{y-K} - \frac{1}{y}\right)dy}_{\ln|y-K| - \ln|y| = \ln\left|\frac{y-K}{y}\right|} = -\frac{r}{K}t + C$$

$$\ln|y-K| - \ln|y| = \ln\left|\frac{y-K}{y}\right| = -rt + C$$

$$\rightsquigarrow \left|\frac{y-K}{y}\right| = e^{-rt+c} = e^{-rt}A \Rightarrow \frac{y-K}{y} = \pm Ae^{-rt}$$

$$\boxed{\frac{y-K}{y} = Ce^{-rt}} \quad (*)$$

Note: $\dfrac{y(0)-K}{y(0)} = Ce^{-r\cdot 0} \Rightarrow C = \boxed{\dfrac{y(0)-K}{y(0)}}$

Can simplify $(*)$: $1 - \dfrac{K}{y} = Ce^{-rt} \rightsquigarrow 1 - Ce^{-rt} = \dfrac{K}{y}$

$$\rightsquigarrow y(t) = \frac{K}{1 - Ce^{-rt}}$$

Note: if $y(0) > K > 0$, then $C > 0$

if $0 < y(0) < K$, then $C > 0$

$r > 0 \Rightarrow e^{-rt} \to 0^+$ as $t \to \infty \Longrightarrow y(t) \to K$

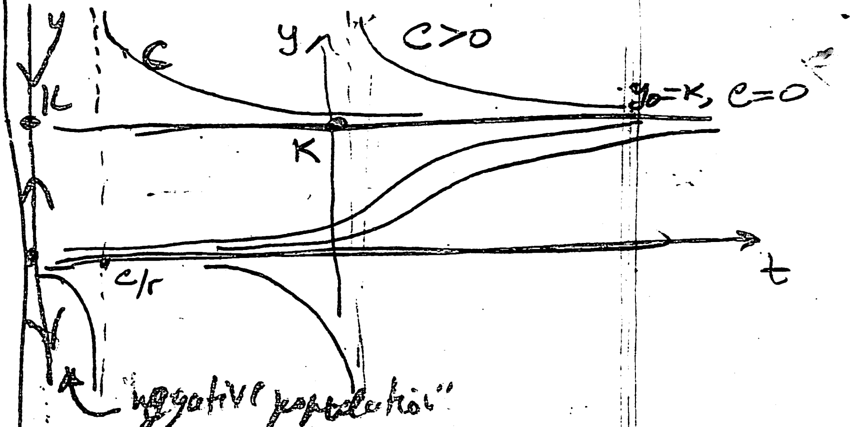
Logistic Growth Formula

The solution to initial-value problem

$$y' = ry\left(1-\frac{y}{K}\right), \quad y(0) = y_0, \ (\neq 0)$$

is $y(t) = \dfrac{K}{1-\frac{y_0-K}{y_0}e^{-rt}}$

Sketch $(K>0, r>0)$



$C = \frac{y(0)-K}{y(0)} > 0 \Leftrightarrow y(0) > K$

$\Rightarrow -Ce^{-rt}$ increases to 0 as $t \to \infty$

$\Rightarrow 1-Ce^{-rt} \nearrow 1$ as $t \to \infty$

$\Rightarrow \frac{K}{1-Ce^{-rt}} \searrow K$ as $t \to \infty$

Not defined if $1-Ce^{-rt} = 0 \Leftrightarrow e^{-rt} = 1/c$

$\Leftrightarrow -rt = \ln 1/c = -\ln c \Rightarrow t = \frac{1}{r}\ln c$

if $t \to -\infty$, $1-Ce^{-rt} \to -\infty$

$\Rightarrow \frac{K}{1-Ce^{-rt}} \nearrow 0$ as $t \to -\infty$

$C = \frac{y(0)-K}{y(0)} < 0 \Leftrightarrow y(0) < K$

$\Rightarrow -Ce^{-rt} \searrow 0$ as $t \to \infty \Rightarrow 1-Ce^{-rt} \searrow 1$

$\nearrow \infty$ as $t \to -\infty \Rightarrow \nearrow r$

$\Rightarrow \frac{K}{1-Ce^{-rt}}$ $\nearrow K$ as $t \to \infty$, $\searrow 0$ as $t \to -\infty$

defined for t since $1-Ce^{-rt} \geq 0$ $(\neq)$ if $C<0 \Leftrightarrow -C>0$

Slope diagram ←
invariant under shifts