MAT 127

02/10/22

3rd board

⇒ HW2 due now ⇐

Solutions on website tomorrow

HW3 due Th, 9/24

Last time: Find exact solutions to

$$y' = \overset{f(x)}{f(x)}g(y), \quad y = y(x)$$

Step 1: $y' = \dfrac{dy}{dx} \longrightarrow \dfrac{dy}{dx} = f(x)g(y)$

Step 2: Separate x and y: $\dfrac{dy}{g(y)} = f(x)\,dx$

Step 3: Integrate $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$

get $\boxed{G(y) = F(x) + C}$ implicitly defines $y = y(x)$

implicitly defines y as a function of x

Step 4: Simplify as much as possible

Step 5: Check if missing constant solutions

$y(x) = y_0$ for all $x \Leftrightarrow g(y_0) = 0$

(Step 6: check "the general solution" is a solution

→ compute y', $f(x)g(y)$, and compare

Today and next time: some applications of this

Example 1: Find an equation for the curve in xy-plane passing thr. (2, 1) and with slope $\dfrac{3x^2}{3y^2+4y}$ at (x,y).

Approach: assume the curve is the graph of $y = y(x)$

$\Rightarrow y'(x) = \text{slope at } (x,y) = \dfrac{3x^2}{3y^2+4y}, \quad y(2) = 1$

∴ get initial value problem:

$$y' = \frac{3x^2}{3y^2+4y}, \quad y(2) = 1$$

Find the general solution to diff. eqn. and then find C

(1) $\dfrac{dy}{dx} = \dfrac{3x^2}{3y^2+4y}$

(2) $(3y^2+4y)\,dy = 3x^2\,dx$

(3) $\displaystyle\int (3y^2+4y)\,dy = \int 3x^2\,dx$

$y^3 + 2y^2 = x^3 + C$ ← general solution

defines $y = y(x)$ implicitly

Plug in $(x,y) = (2,1)$: $\underbrace{1^3 + 2\cdot 1^2}_{3} = 2^3 + C \Rightarrow C = -5$

∴ $\boxed{y^3 + 2y^2 = x^3 - 5}$ is an equation of the curve passing thr. (2, 1) and with slope $\dfrac{3x^2}{3y^2+4y}$ at (x,y)

passes thr. (2, 1) - 4

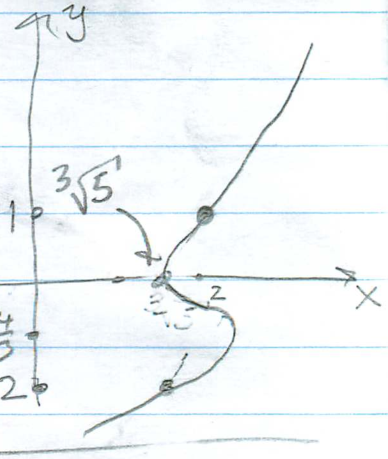
x increases ⇒ y increase

curves vertical if $y' = \infty$

y' if $y = 0, -\frac{4}{3}$

$\Rightarrow \dfrac{dx}{dy} = 0$ if $y = 0$

find $\dfrac{d^2x}{dy^2}$ (>0)



Example 3 (Mixing problem):

salt solution of .5 Kg/100L enters at 10 L/min

water tank

contains 100L of salt solution with 1 Kg of salt initially

flow out = 10L/min of thoroughly mixed solution

Q: How much salt is in the tank after 1 hour?

$y(t)$ = amount of salt in the tank at time t

$y(0) = 1$ (Kg)

$$y'(t) = \underbrace{y'_{in}(t)}_{\frac{10L}{min}\cdot\frac{.5Kg}{100L} = .05\ (Kg/min)} - y'_{out}(t)$$

correct unit

(solution flow) · (salt concentration) = (salt flow)

$$y'_{out}(t) = \underbrace{(\text{solution flow})}_{10L/min} \cdot \underbrace{(\text{salt concentration})}_{\frac{\text{salt in the tank}}{\text{volume in the tank}} = \frac{y(t)}{100}}$$

stays constant

$$\therefore\ y' = \frac{1}{20} - \frac{1}{10}y \iff 20y' = 1-2y$$

$\therefore$ $y(t)$ solves initial-value problem

$$20y' = 1-2y, \quad y(0) = 1$$

Find general solution of diff eq., then C:

$$20\frac{dy}{dt} = 1-2y \rightarrow 20\frac{dy}{1-2y} = dt$$

$$\rightarrow \int 20\frac{dy}{1-2y} = dt \rightarrow \frac{20}{-2}\ln|1-2y| = t + C$$

Simplify: $\ln|1-2y| = -t/10 - C/10 \rightarrow C$

do not erase

$$|1-2y| = e^{-t/10}\, e^{C} \rightarrow A$$

$$1-2y = \pm A e^{-t/10}$$

$$y = \frac{1}{2} \underbrace{\mp\frac{1}{2}A}_{C} e^{-t/10}$$

do not erase

earlier

Find C: $y(0) = \frac{1}{2} + C\underbrace{e^{-0/10}}_{1} = 1 \Rightarrow C = \frac{1}{2}$

$\therefore\ y(t) = \frac{1}{2} + \frac{1}{2}e^{-t/10}$ amount of salt after t mins

$y(1hr) = \frac{1}{2} + \frac{1}{2}e^{-60/10} = \frac{1}{2} + \frac{1}{2}e^{-6} \approx .5012$ Kg

Reality check: approaches concentration of .5 Kg/100L as $t \to \infty$, of incoming solution.

Approach to Mixing Problems

(0) Draw a picture

(1) Take $y(t)$ = amount of "salt" at time t

Find $y(0)$ = usually given

$y' = \dots$ the tricky part

$$\boxed{y' = y'_{in} - y'_{out}}$$

(2) $y'_{in} = \underbrace{(\text{concentration in}) \cdot (\text{flow rate in})}_{\text{usually given}}$

$y'_{out} = \underbrace{(\text{concentration out})}_{y/\text{volume}} \cdot \underbrace{(\text{flow rate out})}_{\text{usually given}}$

red

y/volume !

(3) Solve resulting initial value problem

(4) Plug in new t if asked to find $y(t_{new})$

Careful with units