⇒ HW1 due now ⇐

solutions on website [illegible]

02/03/22

Last time: sketch solution curves for differential equation

$$y' = f(x,y), \quad y = y(x)$$

without solving it

Steps: (1) Compute slopes $y' = f(x,y)$ at lots of points (x_i, y_j)

(2) mark slopes in xy-plane by short line segments

(3) sketch graphs roughly matching the slopes

Main example: $y' = y - x$, $y = y(x)$

Today: Euler's method

if $y = y(x)$ is the solution to initial-value problem

$$y' = f(x,y), \quad y(x_0) = y_0,$$

estimate $y(x_f)$. f = final

~ find points on solution curves

Example: $y' = y - x$, $y = y(x)$, $y(0) = 0$.

Estimate $y(1)$.

1st try: start at $(0,0)$, move along the tangent line to the graph at $(0,0)$.

Slope of tangent line: $y'(0,0) = 0 - 0$

Equation of tangent line:

$$y = y_0 + \text{slope} \cdot (x - x_0) = 0 + 0 \cdot (x - 0) = 0$$

$\therefore$ tangent line horizontal, so

at $x = 1$ on the tangent line $y = 1$

First estimate for $y(1) \approx 0$

solution to $y' = y - x$, $y(0) = 0$

do not erase

Better try: first estimate $y(\frac{1}{2})$

then use this to estimate $y(1)$

$$y(\tfrac{1}{2}) \approx \underbrace{y(0) + y'(0) \cdot (\tfrac{1}{2} - 0)}_{\text{moving along the tangent line at } (0,0)} = 0 = y_1$$

$$y(1) \approx \underbrace{y_1 + (\text{slope at } (x_1, y_1)) \cdot (1 - \tfrac{1}{2})}_{\text{moving along slope at } (x_1, y_1) = (\frac{1}{2}, 0)}$$

with $x_1 = 1/2$

Make a table

do not erase

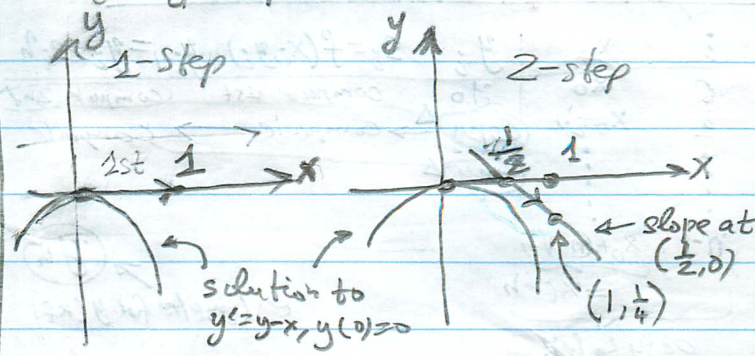
slope at $(\frac{1}{2},0) = 0 - \frac{1}{2} = -\frac{1}{2}$

$y(1) \approx 0 + \frac{1}{2}\cdot(1-\frac{1}{2}) = -\frac{1}{4}$

$\therefore -\frac{1}{4}$ = 2nd try at estimating $y(1)$

solution to $y' = y - x,\ y(0) = 0$

Enlarged picture near (0,0)



4-step estimate of $y(1)$:

size of each $h = \Delta x = \frac{1-0}{4} = 1$

get estimate y_1, y_2, y_3, y_4 for

y at $x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$

(1) $y(\frac{1}{4}) \approx \underbrace{y(0) + (\text{slope at }(0,0))\cdot(\frac{1}{4}-0)}_{\text{moving along the slope}} = 0 + 0\cdot\frac{1}{4} = 0 = y_1$

(2) $y(\frac{1}{2}) \approx y_2 \equiv y_1 + \underbrace{(\text{slope at }(x_1,y_1))}_{0-\frac{1}{4}=-\frac{1}{4}}\cdot\underbrace{(\frac{1}{2}-\frac{1}{4})}_{\frac{1}{4}=h}$

$(x_1,y_1) = (\frac{1}{4}, 0)$

$\Rightarrow y_2 = 0 + (-\frac{1}{4})\cdot\frac{1}{4} = -\frac{1}{16}$ → estimate for $y(\frac{1}{2})$

(3) $y(\frac{3}{4}) \approx y_3 \equiv y_2 + \underbrace{(\text{slope at }(x_2,y_2))}_{-\frac{1}{16}-\frac{1}{2}=-\frac{9}{16}}\cdot\underbrace{(\frac{3}{4}-\frac{1}{2})}_{h}$

$(x_2,y_2) = (\frac{1}{2}, -\frac{1}{16})$

$\Rightarrow y_3 = -\frac{1}{16} + (-\frac{9}{16})\cdot\frac{1}{4} = -\frac{4+9}{64} = -\frac{13}{64}$ → estimate for $y(\frac{3}{4})$

do not erase

(4) $y(1) \approx \cancel{y_4} = y_3 + \underbrace{(\text{slope at }(x_3,y_3))}_{-\frac{13}{64}-\frac{3}{4}=-\frac{13+48}{64}=-\frac{61}{64}}\cdot\underbrace{(1-\frac{3}{4})}_{\frac{1}{4}=h}$

$(x_3,y_3) = (\frac{3}{4}, -\frac{13}{64})$

$\Rightarrow y_4 = -\frac{13}{64} + (-\frac{61}{64})\cdot\frac{1}{4} = -\frac{52+61}{256} = -\frac{113}{256}$

estimate for $y(1)$

Estimates for $y(1)$, where $y' = y - x,\ y(0) = 0$

1-step: $y(1) \approx 0$

2-step: $y(1) \approx -\frac{1}{4} = -.25$

4-step: $y(1) \approx -\frac{113}{256} \approx -.44$

Actual solution: $y(x) = 1 + x - e^x \Rightarrow y(1) = 2 - e^1 \approx -.72$

$\therefore$ estimates get better with more steps

Problem: Suppose $y = y(x)$ solves $y' = f(x,y),\ y(x_0) = y_0$.

Estimate $y(x_f)$ for some $x_f > x_0$.

Euler's method with $n \geq 1$ steps

(0) Step size $h = \frac{x_f - x_0}{n}$

(1) will estimate $y(x_f)$ at

$x_1 = x_0 + h,\ x_2 = x_0 + 2h, \dots x_i = x_0 + hi,\ 1 \leq i \leq n$

(1) s_0 = slope at $(x_0,y_0) = f(x_0,y_0)$

$y(x_1) \approx y_1 \equiv y_0 + s_0 h \rightsquigarrow (h = x_1 - x_0)$

(2) s_1 = slope at $(x_1,y_1) = f(x_1,y_1)$

$y(x_2) \approx y_2 \equiv y_1 + s_1 h \rightsquigarrow (h = x_2 - x_1)$

(3) ... s_i = slope at $(x_i,y_i) = f(x_i,y_i)$

$y_{i+1} = y_i + s_i h \rightsquigarrow (h = x_{i+1} - x_i)$

Stop at $y(x_n) \approx y_n$, $x_n = x_0 + nh = x_f$

over →

To Keep track of this: make a table!

i	x_i	y_i	$s_i=f(x_i,y_i)$	$y_{i+1}=y_i+s_i h$
0	x_0	y_0	compute 1st	compute 2nd
1	x_0+h	copy	compute	compute
⋮	⋮	copy		
$n-1$	$\underbrace{x_0+(n-1)h}_{=x_f-h}$			y_n

easy to fill in

estimate for $y(x_f)$

Table for $y'=y-x$, $y(0)=0$

Estimate $y(1)$ with 4 steps $\Rightarrow h=\frac{4-0}{4}=\frac{1}{4}$

i	x_i	y_i	$s_i=y_i-x_i$	$y_{i+1}=y_i+s_i\cdot\frac{1}{4}$
0	0	0	$0-0=0$	$0+0\cdot\frac{1}{4}=0$
1	$\frac{1}{4}$	0	$0-\frac{1}{4}=-\frac{1}{4}$	$0+(-\frac{1}{4})\cdot\frac{1}{4}=-\frac{1}{16}$
2	$\frac{1}{2}$	$-\frac{1}{16}$	$-\frac{1}{16}-\frac{1}{2}=-\frac{9}{16}$	$-\frac{1}{16}-\frac{9}{64}=-\frac{13}{64}$
3	$\frac{3}{4}$	$-\frac{13}{64}$	$-\frac{13}{64}-\frac{3}{4}=-\frac{61}{64}$	$-\frac{13}{64}-\frac{61}{256}=-\frac{113}{256}$

easy

estimate for $y(1)=y(x_4)$

Example: $y'=y-x$, $y(1)=0$

Use Euler's method to estimate $y(2)$ with 3 steps

$h=\frac{2-1}{3}=\frac{1}{3}$

i	x_i	y_i	$s_i=y_i-x_i$	$y_{i+1}=y_i+s_i\cdot\frac{1}{3}$
0	1	0	$0-1=-1$	$0+(-1)\cdot\frac{1}{3}=-\frac{1}{3}$
1	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}-\frac{4}{3}=-\frac{5}{3}$	$-\frac{1}{3}-\frac{5}{9}=-\frac{8}{9}$
2	$\frac{5}{3}$	$-\frac{8}{9}$	$-\frac{8}{9}-\frac{5}{3}=-\frac{23}{9}$	$-\frac{8}{9}-\frac{23}{27}=-\frac{47}{27}$

easy

∴ Our estimate for $y(2)=-\frac{47}{27}\approx-1.74$

Actual solution: $y(x)=1+x-2e^{x-1}$

$\Rightarrow y(2)=2-2e^{1}\approx-3.44$

not good b/c slopes drop quickly and h is too small $\Leftrightarrow \underset{=\Delta x}{h}$ is too large