MAT 127

⇒ HW1 due Th ⇐
by 2:20pm

02/01/22

Last week: differential equations
= equations for function(s) involving their derivatives:

$xy' = y(y-1)$, $y = y(x)$

$y' = y - x$ $y = y(x)$

Aim: to find/describe solution $y = y(x)$

Used: $y' > 0 \Leftrightarrow y$ increasing
$y' < 0 \Leftrightarrow y$ decreasing
$y' = 0 \Leftrightarrow y = \text{const}$

Today: get more accurate description using
$y'(x_0)$ = slope of tangent line to the graph of $y = y(x)$ at $(x_0, y(x_0))$

Example: sketch graphs of solutions of

$$y' = y - x, \quad y = y(x)$$

Plan: (1) compute slopes $y' = y - x$ at various (x, y)
(2) mark slopes in xy-plane
(3) sketch graphs roughly matching the slopes

(1) Slopes for $y' = y - x$:

$y \backslash x$	-2	-1	0	1	2
2	4	3	2	1	0
1	3	2	1	0	-1
0	2	1	0	-1	-2
-1	1	0	-1	-2	-3
-2	0	-1	-2	-3	-4

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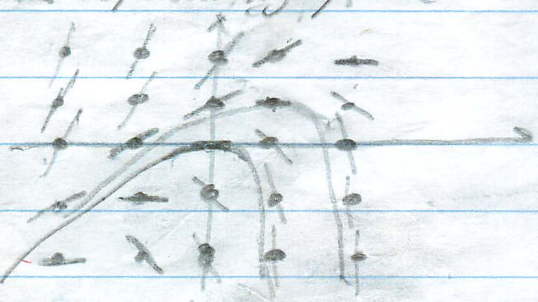
(2) Mark slopes on xy-plane



(3) Put roughly solution curves to match slopes

Graph of solution through (0,0):
- can't go up from (0,0) ⇒ reaches peak at (0,0)
- descends to the left of line $y = x$ as $x \to -\infty$ roughly parallel to $y = x \Rightarrow y = x + 1$
- descends as $x \to \infty$, very rapidly

(actual solution $y = 1 + x - e^x$ satisfies all these)

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Graph of solution through (1,1):
can't go up from (1,1) ⇒ reaches peak at (1,1)
- descends to the left of line $y = x$ as $x \to -\infty$ roughly parallel to $y = x \Rightarrow y = x + 1$
- descends as $x \to \infty$, very rapidly

(actual solution $y = 1 + x - e^{x-1}$)

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Graph of solutions thr (0,1)

look like straight line $y = 1 + x$

Check if this is a solution:

$y' = 1, \quad y - x = 1 \Rightarrow y' = y - x$

Other solutions: rise

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Graphs of other solutions

Roughly parallel to $y = x$ as $x \to -\infty$

$\Rightarrow \to y = x + 1$

rise very rapidly as $x \to \infty$

or drop

do not erase

General solution: $y = 1 + x + Ce^{x}$

Easy part to check: each of these is a solution

$y' = 1 + Ce^{x}, \quad y - x = 1 + Ce^{x} \Rightarrow y' = y - x$ ✓

Hard part to check: if $y = y(x)$ satisfies $y' = y - x$,

then $y(x) = 1 + x + Ce^{x}$ for some constant C

(can do using HW2 - Problem B on HW2)

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Consistent with graph:

$C = 0$: $y = x + 1$ is a solution

$C > 0$: $y = 1 + x + Ce^{x}$ rise rapidly as $x \to \infty$

approaches $y = x + 1$ as $x \to -\infty$

$C < 0$: $y = 1 + x + Ce^{x}$ drops rapidly as $x \to \infty$

approaches $y = x + 1$ as $x \to -\infty$

over $\Longrightarrow$

"Diagrams of slopes" are called "direction fields"

(1) obtained by computing : $y' = f(x, y)$
at many points (x_i, y_i)

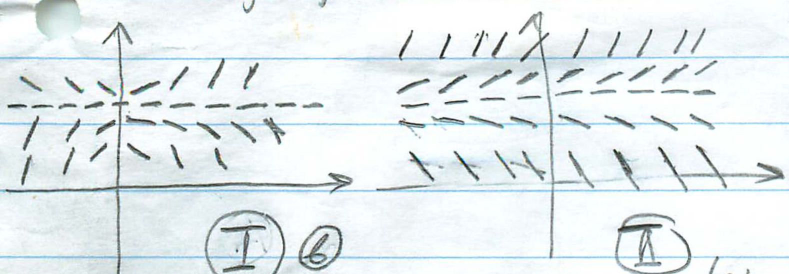
(2) indicate graphs of solutions

More examples of direction fields

next page

shift by one

More examples of direction fields



I (b) $y' = x(y-2)$

II $y' = y-2$ (a)

These correspond to to

(a) $y' = y-2$ (b) $y' = x(y-2)$

Which is which?

$\mathrm{I} \longleftrightarrow$ (b), $\mathrm{II} \longleftrightarrow$ (a)

Why?

- $\underbrace{y' = y-2 > 0}_{\text{slopes up}}$ if $y>2$ no matter what x is

  slopes up / (not \\)

  yes in II, but not in I $\Rightarrow$ (a) $\not\longleftrightarrow$ I

- $y' = x(y-2) > 0$ if $y>2, x>0$ or $y<2, x<0$

  yes in I but not in II

  $\Rightarrow$ (b) $\not\longleftrightarrow$ II

Some solution curves:

(0) $y=2$ is a solution in both cases

In I = (b): ~~for y>2 graphs reach~~

for $y>2$: graphs reach lowest point on y-axis
rise rapidly on both sides

for $y<2$: graphs reach highest point on y-axis
drop rapidly on both sides

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II - (a):

for $y>2$: graphs rise, first slowly, then rapidly

for $y<2$: graphs fall, — " —

"picture symmetry in horizontal direction

$\Rightarrow$ horizontal shifts of graphs of solutions are still graphs of solutions:

$y = y(x)$ is a solution $\Rightarrow$ $\tilde{y}(x) = y(x-a)$ is also a solution

$\Rightarrow$ happens only for autonomous equations

Actual solutions

$y' = x(y-2) \longrightarrow y(x) = 2 + Ce^{x^2/2}$

$y' = y-2 \longrightarrow y = 2 + Ce^{x}$

(next week)

expected behaviour