MAT 127

01/27/22

⇒ HW1 on website ⇐ due on Th 2/3, before

Last time: (1) mini-reviews of derivatives

(2) examples of differential equations, their solutions

properties of solutions (can describe without knowing solutions)

modelling by diff. equations

A first-order differential equation is an equation involving a function and its derivative

Examples last time:

(1) Exponential growth model: $y' = ry$, $y = y(t)$, $r = \text{const}$

(2) Logistic growth model: $y' = ry\left(1 - \frac{y}{K}\right)$, $y = y(t)$, $r, K = \text{const}.$

Other examples: (3) $y' + (\tan x)y = \cos^2 x$, $y = y(x)$

(4) $xy' = -y^2 + x$, $y = y(x)$

(5) $y' + x^2 = 0$, $y = y(x)$

A solution of a differential equation is a function that satisfies the equation

Examples last time:

(1) Equation: $y' = ry$, $y = y(t)$

a solution: $y(t) = 0$ for all t

another solution: $y(t) = 5e^{rt}$

The general solution of a differential equation is the set of all solutions of the equations

Example (1): $y' = ry$, $y = y(t)$

general solution: $y(t) = Ce^{rt}$, C = any constant

$\Leftrightarrow$ (i) $y(t) = Ce^{rt}$ is a solution of diff. eqn. for every fixed constant C (checked last time)

(ii) if $f = f(t)$ solves the diff. eqn. $y' = ry$, $y = y(t)$, then $f(t) = Ce^{rt}$ for some constant C (harder, will check later)

Example 2: $y' = x^2$, $y = y(x)$

Fundamental Theorem of Calculus $\Rightarrow y(x) = \int x^2\,dx = \frac{1}{3}x^3 + C$

$\therefore$ $y(x) = \frac{1}{3}x^3 + C$ is the general solution of this equation

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To check that a function f solves diff. eqn:

(1) Compute f'

(2) Plug in f, f' into diff. equation to see if equality holds.

Example: $xy' + y = 2x$, $y = y(x)$

Check that $y(x) = x - \frac{1}{x}$ is a solution

(1) $y'(x) = 1 - (x^{-1})' = 1 - (-1)\cdot x^{-2} = 1 + x^{-2}$

$xy'(x) + y(x) = x(1 + x^{-2}) + (x - x^{-1}) = 2x$ ✓

$\therefore$ $y(x) = x - \frac{1}{x}$ satisfies $xy' + y = 2x$

Sometimes: given differential equation for $y=y(t)$ and starting-value $y(t_0)=y_0$, want to find/describe $y(t)$

Example: $y'=y(y-1)$, $y=y(t)$, $y(0)=2$

(a) Show that $y(t)=\frac{1}{1-Ce^t}$ is a solution of diff. eqn. for every value of C

(b) Find a solution to the initial value problem

(a) $y(t)=\frac{1}{1-Ce^t} = (1-Ce^t)^{-1}$

$y'(t) = -\frac{1}{(1-Ce^t)^2}\cdot(-Ce^t) = \frac{Ce^t}{(1-Ce^t)^2}$

$y(t)=y(y-1) = \frac{1}{1-Ce^t}\cdot\underbrace{\left(\frac{1}{1-Ce^t}-1\right)}_{\frac{Ce^t}{1-Ce^t}} = \frac{Ce^t}{(1-Ce^t)^2}$

$\therefore y(t)=\frac{1}{1-Ce^t}$ solves $y'=y(y-1)$

(b) $y(t)=\frac{1}{1-Ce^t}$ satisfies diff. eqn. for every fixed C

$\Rightarrow$ enough to find C s.t. $y(0)=2$

$y(0)=\frac{1}{1-C\cdot e^0} = \frac{1}{1-C} = 2 \Rightarrow C=\frac{1}{2}$

$\therefore y(t)=\frac{1}{1-\frac{1}{2}e^t}$ solves the initial-value problem

~~For~~ what t is $y(t)=\frac{1}{1-\frac{1}{2}e^t}$ defined? (not!)

$y(t)$ not defined at $1-\frac{1}{2}e^t=0 \Leftrightarrow e^t=2$

$\Leftrightarrow t=\ln 2$

$\Rightarrow y(t)=\frac{1}{1-\frac{1}{2}e^t}$ is defined on $\underbrace{(-\infty,\ln 2)}_{\text{this contains } 0}$, $(\ln 2,\infty)$

initial value of t_0

To check that a function f defined on (a,b) solves given initial value problem:

(1) check that $f(t)$ solves the diff. eqn. (compute $f'(t)$; plug in $f(t)$, $f'(t)$ into eqn.)

(2) check that $t_0\in(a,b)$ and $f(t_0)$ is as required

(3) check that f and all functions in diff. eqn. are defined on (a,b)

(4) (a,b) can't be taken any larger: f is not defined at a or b or another function in diff. eqn. not defined at a or b

(HW: 1.2, #7)

Last time: can describe/sketch solutions of diff. eqn. without solving it

Used: $y'>0 \Leftrightarrow y$ increases

$y'<0 \Leftrightarrow y$ decreases

$y'=0 \Leftrightarrow y=\text{const}$

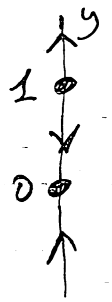
Example: $y'=y(y-1)$, $y=y(x)$

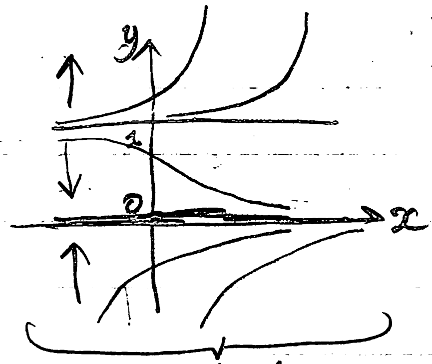
y increases if $y>1$ or $y<0$

y decreases if $0<y<1$

$y(t)=0, 1$ for all t are constant solutions

over $\Longrightarrow$





groups of solutions

Qualitative properties depend only on y, not x

b/c $y' = y(y-1)$

is autonomous first-order diff. eqn

↪ no explicit dependence on x in equation

↪ e.g. not $y' = y(y-1)+x^2$

1st diagram suitable only for autonomous

Can get more accurate pictures using:

$y'(x_0)$ = slope of the tangent line to the graph of y at $(x_0, y(x_0))$

Example: $y' = y - x$, $y = y(x)$

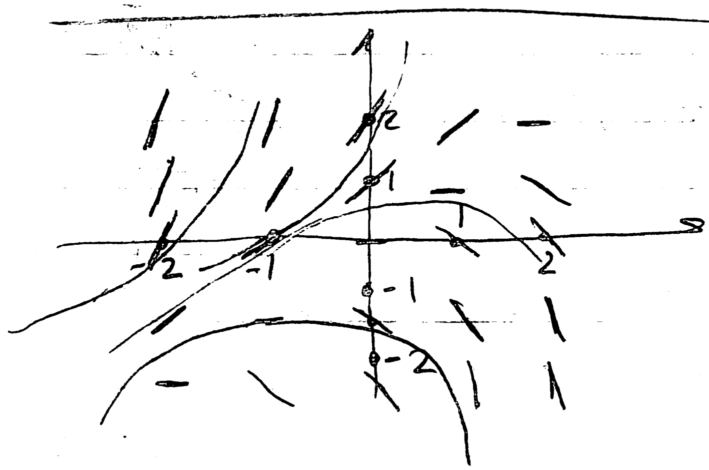
slope at $(0,0) \to 0-0$

$(0,1) \to 1-0$

Slopes for $y' = y - x$

$y \backslash x$	-2	-1	0	1	2
-2	0	-1	-2	-3	-4
-1	1	0	-1	-2	-3
0	2	1	0	-1	-2
1	3	2	1	0	-1
2	4	3	2	1	0

slope at $(-2,1)$



1. Mark slopes at various points
2. Draw a graph roughly tangent to these
3. This approximates solution

Actual solutions: $y = x + 1 + Ce^{x}$ $\qquad y - x = 1 + Ce^{x}$

$y' = 1 + Ce^{x}$